Journal of Functional Analysis 266 (2014) 5439–5447

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

The strong approximation property and the weak bounded approximation property

Ju Myung Kim [a,1], Bentuo Zheng [b,*,2]

[a] Department of Mathematical Sciences, Seoul National University, Seoul 151-742, Republic of Korea
[b] Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152-3240, United States

ARTICLE INFO

Article history:
Received 2 October 2013
Accepted 17 December 2013
Available online 6 January 2014
Communicated by G. Schechtman

Keywords:
Approximation property
Strong approximation property
Weak bounded approximation property
Bounded approximation property

ABSTRACT

We show that the strong approximation property (strong AP) (respectively, strong CAP) and the weak bounded approximation property (respectively, weak BCAP) are equivalent for every Banach space. This gives a negative answer to Oja's conjecture. As a consequence, we show that each of the spaces c_0 and ℓ_1 has a subspace which has the AP but fails to have the strong AP.

1. Introduction and the main results

One of the most important properties in Banach space theory is the *approximation property* (AP) which was systematically investigated by Grothendieck [7]. A Banach space X is said to have the AP if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank continuous linear map (operator) S on X such that $\sup_{x\in K} \|Sx - x\| \leqslant \varepsilon$, briefly, $id_X \in \overline{\mathcal{F}(X)}^{\tau_c}$, where id_X is the identity map on X, $\mathcal{F}(X)$

* Corresponding author.
E-mail addresses: kjm21@kaist.ac.kr (J.M. Kim), bzheng@memphis.edu (B. Zheng).

[1] Ju Myung Kim's research was supported by NRF-2013R1A1A2A10058087 funded by the Korean Government.
[2] Bentuo Zheng's research was supported in part by NSF grant DMS-1200370.

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.12.016

is the space of all finite rank operators on X and τ_c is the compact-open topology on the space $\mathcal{L}$ of all operators between Banach spaces. If $id_X \in \overline{\{S \in \mathcal{F}(X)\colon \|S\| \leqslant \lambda\}}^{\tau_c}$ for $\lambda \geqslant 1$, then we say that X has the *λ-bounded approximation property* (λ-BAP).

Oja [13] introduced a stronger form of the AP. A Banach space X is said to have the *strong approximation property* (strong AP) if for every separable reflexive Banach space Y and every $R \in \mathcal{K}(X,Y)$, the space of all compact operators from X into Y, there exists a bounded net (U_α) in $\mathcal{F}(X,Y)$ such that $\|U_\alpha x - Rx\| \to 0$ for every $x \in X$, which is equivalent to that $R \in \overline{\{U \in \mathcal{F}(X,Y)\colon \|U\| \leqslant \lambda_R\}}^{\tau_c}$ for some $\lambda_R > 0$ because the compact-open topology and the strong operator topology on $\mathcal{L}$ coincide on any bounded subset of $\mathcal{L}$. This notion is actually stronger than the AP (see [13, Theorem 2.1] or [8]).

Lima and Oja [11] introduced and investigated a weaker form of the BAP. For $\lambda \geqslant 1$, a Banach space X is said to have the *weak λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and every $R \in \mathcal{W}(X,Y)$, the space of all weakly compact operators from X into Y, we have $id_X \in \overline{\{S \in \mathcal{F}(X)\colon \|RS\| \leqslant \lambda\|R\|\}}^{\tau_c}$. The formal implications between these approximation properties are:

$$\lambda\text{-BAP} \Longrightarrow \text{ weak } \lambda\text{-BAP} \Longrightarrow \text{ strong AP } \Longrightarrow \text{AP}.$$

A long standing open problem in this direction is whether the BAP and the AP are equivalent for dual spaces. Grothendieck [7] proved that the 1-BAP and the AP are equivalent for every separable dual space or every reflexive Banach space (see, e.g., [1, Theorems 3.6 and 3.7]). Although a general answer to this problem is not known, some partial results were obtained. It was shown in [11, Theorem 3.6] that the weak 1-BAP and the AP are equivalent for every dual space and it was shown in [12, Corollary 1] that the λ-BAP and the weak λ-BAP are equivalent for every Banach space with separable dual. In this paper, we show in Corollary 1.2 that the BAP and the strong AP are equivalent for every Banach space whose dual space is separable.

Lima and Oja [11] conjectured that the weak BAP and the BAP are different properties and Oja [13, Conjectures 3.5 and 3.7] conjectured that the strong AP and the weak BAP are different properties. We show in Theorem 1.3 that the strong AP and the weak BAP are actually equivalent for every Banach space. This gives a negative answer to Oja's conjecture.

Figiel and Johnson [5] first constructed a Banach space X_{FJ} having the AP but failing to have the BAP. Moreover, its dual space is separable, hence it follows from Corollary 1.2 that the Banach space X_{FJ} does fail to have the strong AP. As a consequence, we show that each of the spaces c_0 and ℓ_1 has a subspace which has the AP but fails the strong AP. It also follows that if X has the AP but its dual space fails to have the AP, then $\ell_p(X)$ $(1 \leqslant p < \infty)$ has a subspace which has the AP but fails to have the strong AP, where $\ell_p(X)$ is the Banach space of all absolutely p-summable sequences in X.

The main results. We proceed on a more general setting to encompass other approximation properties. Let $T \in \mathcal{L}(X)$ and let $\mathcal{A}(X)$ be a convex subset of $\mathcal{L}(X)$. A Banach