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Semilinear fractional elliptic equations with gradient nonlinearity involving measures

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Abstract

We study the existence of solutions to the fractional elliptic equation (E1) $(-\Delta)^\alpha u + \epsilon g(|\nabla u|) = \nu$ in an open bounded regular domain Ω of $\mathbb{R}^N$ $(N \geqslant 2)$, subject to the condition (E2) $u = 0$ in Ω^c, where $\epsilon = 1$ or -1, $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha \in (1/2, 1)$, ν is a Radon measure and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function. We prove the existence of weak solutions for problem (E1)–(E2) when g is subcritical. Furthermore, the asymptotic behavior and uniqueness of solutions are described when $\epsilon = 1$, ν is Dirac mass and $g(s) = s^p$ with $p \in (0, \frac{N}{N-2\alpha+1})$.

Keywords: Fractional Laplacian; Radon measure; Green kernel; Dirac mass

Contents

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) be an open bounded C^2 domain and $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function. The purpose of this paper is to study the existence of weak solutions to the semilinear fractional elliptic problem with $\alpha \in (1/2, 1)$,

$$\begin{aligned} &(-\Delta)^\alpha u + \epsilon g\big(|\nabla u|\big) = \nu \quad && \text{in } \Omega, \\ &u = 0 && \text{in } \Omega^c, \end{aligned} \tag{1.1}$$

where $\epsilon = 1$ or -1 and $\nu \in \mathfrak{M}(\Omega, \rho^\beta)$ with $\beta \in [0, 2\alpha - 1)$. Here $\rho(x) = \operatorname{dist}(x, \Omega^c)$ and $\mathfrak{M}(\Omega, \rho^\beta)$ is the space of Radon measures in Ω satisfying

$$\int_\Omega \rho^\beta d|\nu| < +\infty. \tag{1.2}$$

In particular, we denote $\mathfrak{M}^b(\Omega) = \mathfrak{M}(\Omega, \rho^0)$. The associated positive cones are respectively $\mathfrak{M}_+(\Omega, \rho^\beta)$ and $\mathfrak{M}_+^b(\Omega)$. According to the value of ϵ, we speak of an absorbing nonlinearity in the case $\epsilon = 1$ and a source nonlinearity in the case $\epsilon = -1$. The operator $(-\Delta)^\alpha$ is the fractional Laplacian defined as

$$(-\Delta)^\alpha u(x) = \lim_{\varepsilon \to 0^+} (-\Delta)_\varepsilon^\alpha u(x),$$

where for $\varepsilon > 0$,

$$(-\Delta)_\varepsilon^\alpha u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\varepsilon\big(|x - z|\big)\, dz \tag{1.3}$$

and

$$\chi_\varepsilon(t) = \begin{cases} 0, & \text{if } t \in [0, \varepsilon], \\ 1, & \text{if } t > \varepsilon. \end{cases}$$

In a pioneering work, Brezis [7] (also see Bénilan and Brezis [1]) studied the existence and uniqueness of the solution to the semilinear Dirichlet elliptic problem

$$\begin{aligned} &-\Delta u + h(u) = \nu \quad && \text{in } \Omega, \\ &u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$