Journal of Functional Analysis 268 (2015) 2586–2620

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Free transport for finite depth subfactor planar algebras ☆

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A R T I C L E   I N F O

Article history:
Received 27 June 2014
Accepted 19 December 2014
Available online 27 January 2015
Communicated by S. Vaes

Keywords:
Free probability
Subfactors
Planar algebras
Free transport

A B S T R A C T

Given a finite depth subfactor planar algebra $\mathcal{P}$ endowed with the graded $*$-algebra structures $\{Gr_k^+\mathcal{P}\}_{k\in\mathbb{N}}$ of Guionnet, Jones, and Shlyakhtenko, there is a sequence of canonical traces $Tr_{k,+}$ on $Gr_k^+\mathcal{P}$ induced by the Temperley–Lieb diagrams and a sequence of trace-preserving embeddings into the bounded operators on a Hilbert space. Via these embeddings the $*$-algebras $\{Gr_k^+\mathcal{P}\}_{k\in\mathbb{N}}$ generate a tower of non-commutative probability spaces $\{M_{k,+}\}_{k\in\mathbb{N}}$ whose inclusions recover $\mathcal{P}$ as its standard invariant. We show that traces $Tr_{k,+}^{(v)}$ induced by certain small perturbations of the Temperley–Lieb diagrams yield trace-preserving embeddings of $Gr_k^+\mathcal{P}$ that generate the same tower $\{M_{k,+}\}_{k\in\mathbb{N}}$.

1. Introduction

Despite the relatively innocuous definition of a subfactor, Jones showed in [6–8] that there is in fact an incredibly rich structure underlying the inclusion of one II_1 factor in another. In particular, one can associate to a subfactor $N \subset M$ its standard invariant: a planar algebra. It was later shown by Popa in [12] that in fact every subfactor planar algebra can be realized through this association.

☆ Research supported by NSF grants DMS-1161411 and DMS-0838680.
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http://dx.doi.org/10.1016/j.jfa.2014.12.018

In [2] Guionnet, Jones, and Shlyakhtenko produce an alternate proof of this fact by constructing the subfactors via free probabilistic methods. Given a subfactor planar algebra $\mathcal{P}$, for each $k \geq 0$ one can turn $Gr_k^+ \mathcal{P} = \bigoplus_{n \geq k} \mathcal{P}_{n,+}$ into a $*$-algebra with a trace $Tr_{k,+}$ defined by a particular pairing with Temperley–Lieb diagrams. Then each $Gr_k^+ \mathcal{P}$ embeds into the bounded operators on a Hilbert space and generates a II_1 factor $M_{k,+}$. Moreover, one can define inclusion maps $i_k^{k-1}: M_{k-1,+} \to M_{k,+}$ so that the standard invariant associated to the subfactor inclusion $i_k^{k-1}(M_{k-1,+}) \subset M_{k,+}$ (for any $k \geq 1$) recovers $\mathcal{P}$ as its standard invariant. The embedding relies on the fact that a subfactor planar algebra $\mathcal{P}$ always embeds into the planar algebra of a bipartite graph $\mathcal{P}^\Gamma$ (*cf.* [8–10]).

It turns out that $Gr_0^+ \mathcal{P}$ embeds as a subalgebra of a free Araki–Woods factor. Free Araki–Woods factors and their associated free quasi-free states, studied by Shlyakhtenko in [13], are type III_λ factors, $0 < \lambda \leq 1$, and can be thought of as the non-tracial analogues of the free group factors. They are constructed starting from a strongly continuous one-parameter group of orthogonal transformations $\{U_t\}_{t \in \mathbb{R}}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$. When $U_t = 1$ for all t, this construction simply yields the free group factor $L(\mathbb{F}_{\dim \mathcal{H}_\mathbb{R}})$. Stone's theorem guarantees the existence of a positive, non-singular generator A satisfying $A^{it} = U_t$ for all $t \in \mathbb{R}$. It was shown in [13] that the type classification of a free Araki–Woods factor is determined by the spectrum of the generator A. Moreover, the action of the modular automorphism group is well known and also depends explicitly on A. In [11], by adapting the free transport methods of Guionnet and Shlyakhtenko (*cf.* [4]), it was shown that non-commutative random variables whose joint law is "close" to a free quasi-free state in fact generate a free Araki–Woods factor. In particular, the finitely generated q-deformed Araki–Woods algebras were shown to be isomorphic to the free Araki–Woods factor for small $|q|$. In this paper we show that the free transport machinery can be encoded via planar tangles and provide an application of free transport to finite depth subfactor planar algebras.

Let $\mathcal{P}$ be a finite depth subfactor planar algebra and $Tr: \mathcal{P} \to \mathbb{C}$ be the state induced by the Temperley–Lieb diagrams via duality. By using the transport construction methods of [11], we show that we can perturb the embedding constructed in [2] to make it state-preserving for states on $\mathcal{P}$ which are "close" to Tr. Moreover, the von Neumann algebra generated by the subfactor planar algebra via this embedding is unchanged. In this context, if $\mathcal{P}$ embeds into $\mathcal{P}^\Gamma$ and μ is the Perron–Frobenius eigenvector for the bipartite graph Γ, then the generator A associated to the free Araki–Woods algebra will be determined by μ.

The free transport methods in [4] and [11] apply only to joint laws of finitely many non-commutative random variables. Since each edge in the graph Γ will correspond to a non-commutative random variable, we can only consider finite depth subfactor planar algebras with these methods.