Journal of Functional Analysis 268 (2015) 2621–2646

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

# Fiber dimension for invariant subspaces

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A R T I C L E I N F O

*Article history:*
Received 2 July 2014
Accepted 25 January 2015
Available online 4 February 2015
Communicated by S. Vaes

*MSC:*
47A53
47A15

*Keywords:*
Analytic operator
Samuel multiplicity
Fiber dimension
Invariant subspace

A B S T R A C T

In this paper we study the fiber dimension of invariant subspaces for a large class of operators. We define a class of invariant subspaces called CF subspaces which are related to the codimension-one property. We obtain several characterizations of CF subspaces, including one in terms of Samuel multiplicity.

Other new findings include: (1) a lattice-additive formula and its applications (Section 4); (2) a new concept of "absorbance" which describes a rough containment relation for invariant subspaces (Section 5); (3) the existence of a unique, smallest CF subspace containing an arbitrary invariant subspace and preserving the fiber dimension (Section 6).



**Contents**



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http://dx.doi.org/10.1016/j.jfa.2015.01.017

## 1. Introduction to CF representations

For a given linear space $\mathcal{M}$ consisting of analytic, $\mathbb{C}^N$-valued functions ($N \in \mathbb{N}$) over a domain $\Omega \subseteq \mathbb{C}$, the *fiber dimension* of $\mathcal{M}$ is defined by

$$\mathrm{fd}(\mathcal{M}) = \sup_{\lambda \in \Omega} \dim \mathcal{M}(\lambda), \tag{1}$$

where the fiber space $\mathcal{M}(\lambda)$ at $\lambda$ is given by

$$\mathcal{M}(\lambda) = \{ f(\lambda): \ f \in \mathcal{M} \} \subseteq \mathbb{C}^N.$$

A point $\lambda$ in $\Omega$ is called a maximal point, or an $m$-point for short, for $\mathcal{M}$ if $\dim \mathcal{M}(\lambda) = \mathrm{fd}(\mathcal{M})$, and is called a *degenerate point* if $\dim \mathcal{M}(\lambda) < \mathrm{fd}(\mathcal{M})$. It is not hard to see that the collection of degenerate points forms a discrete subset in $\Omega$ whose Lebesgue area measure is 0. The set of $m$-points and degenerate points of $\mathcal{M}$ will be denoted by $\mathrm{mp}(\mathcal{M})$ and $\mathcal{Z}_{dg}(\mathcal{M})$, respectively. The fiber dimension has proved to be a fruitful tool to several problems in operator theory in recent years: To the notorious transitive algebra problem [4], to the cellular indecomposable property [3], to multi-variable Fredholm index [14], to Samuel multiplicity [10,11], to general structure of invariant subspaces [12], etc.

In this paper we fix $\Omega$ to be an open, connected, and bounded subset in the complex plane $\mathbb{C}$. Moreover, for convenience, we assume $0 \in \Omega$. We also fix $n, N \in \mathbb{N}$. We denote by $\mathcal{A}_n(\Omega)$ the collection of *analytic operators* which are defined to be the adjoints of operators in the Cowen–Douglas class $\mathcal{B}_n(\Omega^*)$ [5], where $\Omega^* = \{\bar{z}: \ z \in \Omega\}$. By well known constructions in operator theory [6,21], any $T \in \mathcal{A}_n(\Omega)$ can be represented (in the sense of unitary equivalence) as the coordinate multiplication operator $M_z$ on a Hilbert space $H$ satisfying the following:

(1.1) $H$ consists of $\mathbb{C}^N$-valued analytic functions over the domain $\Omega$;
(1.2) The evaluation functional at $\lambda$: $f \in H \to f(\lambda) \in \mathbb{C}^N$ is a continuous map from $H$ to $\mathbb{C}^N$ for each $\lambda \in \Omega$;
(1.3) If $f \in H$, then so is $zf$, where $z$ is the coordinate function; moreover, the multiplication operator $M_{z-\lambda}$ is bounded below for each $\lambda \in \Omega$;
(1.4) $H$ satisfies the condition $\mathrm{cod}(H) = \mathrm{fd}(H)$, where $\mathrm{cod}(H) = \dim(H \ominus zH)$.