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Sharp inequalities over the unit polydisc

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ABSTRACT

Motivated by some results due to Burbea we prove that if a certain sharp integral inequality holds for functions in the unit polydisc which belong to concrete Hardy spaces, then it also holds, in an appropriate form, in the case of functions from arbitrary Hardy spaces. We also examine the equality case. We present an application of this main result to a Burbea inequality which includes an isoperimetric type inequality as a special case.

1. Introduction and the main theorem

1.1. Introduction

In this paper we are interested in a certain kind of integral inequalities for analytic functions in Hardy spaces in the unit polydisc. One of such inequalities is contained in

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a theorem due to Burbea which is formulated below. Before formulation we recall what the generalized Hardy spaces stand for. We recall the definition of classical Hardy spaces in the unit polydisc in the next section.

Introduce firstly the basic notations which will be used. Let $\mathbf{C}$ be the complex plane. Denote by $\mathbf{U}$ the open unit disc, i.e., the set $\{z \in \mathbf{C} : |z| < 1\}$. For an integer $n \geq 1$ let $\mathbf{C}^n$ stand for the n-dimensional complex vector space. The direct product $\mathbf{U}^n = \underbrace{\mathbf{U} \times \cdots \times \mathbf{U}}_{n}$ is the unit polydisc, and $\mathbf{T}^n = \underbrace{\mathbf{T} \times \cdots \times \mathbf{T}}_{n}$ is the unit torus in $\mathbf{C}^n$.

Let $\mathbf{Z}_+$ be the set of all non-negative integers. Denote by $\mathbf{Z}_+^n = \underbrace{\mathbf{Z}_+ \times \cdots \times \mathbf{Z}_+}_{n}$ the set of all multi-indexes. For any complex number q the shifted factorial (the Pochhammer symbol) is

$$(q)_\beta = \begin{cases} q(q+1)\cdots(q+\beta-1), & \text{if } \beta \geq 1, \\ 1, & \text{if } \beta = 0, \end{cases}$$

where β is an integer in $\mathbf{Z}_+$. One may extend this definition as follows. For every $q = (q_1, \dots, q_j, \dots, q_n) \in \mathbf{C}^n$ and $\alpha = (\alpha_1, \dots, \alpha_j, \dots, \alpha_n) \in \mathbf{Z}_+^n$ denote

$$(q)_\alpha = \prod_{j=1}^{n} (q_j)_{\alpha_j}.$$

Denote by $\mathbf{R}_+^n$ the set $\{(x_1, \dots, x_j, \dots, x_m) \in \mathbf{R}^n : x_j \geq 0,\, j = 1, \dots, n\}$.

For $q = (q_1, \dots, q_n) \in \mathbf{R}_+^n \backslash \{0\}$ the generalized Hardy space in the unit polydisc, denoted by $H_q(\mathbf{U}^n)$, is the space of all analytic functions f in $\mathbf{U}^n$ for which the following norm is finite

$$\|f\|_q^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\alpha!}{(q)_\alpha} |a_\alpha|^2,$$

where $a_\alpha = a_\alpha(f)$, $\alpha \in \mathbf{Z}_+^n$ is the α-coefficient in the Taylor expansion for f, i.e., $f(z) = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha z^\alpha$. The space $H_q(\mathbf{U}^n)$ is a reproducing kernel Hilbert space with the kernel

$$K_q(z, w) = \prod_{j=1}^{n} \frac{1}{(1 - z_j \overline{w}_j)^{q_j}},$$

$z = (z_1, \dots, z_j, \dots, z_n) \in \mathbf{U}^n$, $w = (w_1, \dots, w_j, \dots, w_n) \in \mathbf{U}^n$. The details of the construction of generalized Hardy spaces, based on some facts from theory of reproducing kernels, may be found in the Burbea paper [5] in the case of the unit disc; the case of the unit polydisc is given in [6]. For the theory of reproducing kernels we refer to the work of Aronszajn [1].