Journal of Functional Analysis 268 (2015) 2707–2734

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Linear graph transformations on spaces of analytic functions ☆

Alexandru Aleman [a], Karl-Mikael Perfekt [b], Stefan Richter [c], Carl Sundberg [c]

[a] Department of Mathematics, Lund University, Lund, Sweden
[b] Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway
[c] Department of Mathematics, University of Tennessee, Knoxville, TN 37996, USA

ARTICLE INFO

Article history:
Received 20 August 2014
Accepted 19 January 2015
Available online 2 February 2015
Communicated by S. Vaes

MSC:
primary 47A15, 47B32
secondary 47B38

Keywords:
Transitive algebras
Invariant subspaces
Bergman space

ABSTRACT

Let $\mathcal{H}$ be a Hilbert space of analytic functions with multiplier algebra $\mathcal{M}(\mathcal{H})$, and let

$$\mathcal{M} = \{(f, T_1 f, \dots, T_{n-1} f) : f \in \mathcal{D}\}$$

be an invariant graph subspace for $\mathcal{M}(\mathcal{H})^{(n)}$. Here $n \geq 2$, $\mathcal{D} \subseteq \mathcal{H}$ is a vector-subspace, $T_i : \mathcal{D} \to \mathcal{H}$ are linear transformations that commute with each multiplication operator $M_\varphi \in \mathcal{M}(\mathcal{H})$, and $\mathcal{M}$ is closed in $\mathcal{H}^{(n)}$. In this paper we investigate the existence of non-trivial common invariant subspaces of operator algebras of the type

$$\mathcal{A}_\mathcal{M} = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D} : AT_i f = T_i A f \ \forall f \in \mathcal{D}\}.$$

In particular, for the Bergman space L^2_a we exhibit examples of invariant graph subspaces of fiber dimension 2 such that $\mathcal{A}_\mathcal{M}$ does not have any nontrivial invariant subspaces that

☆ A part of this work was completed while the second author visited the University of Tennessee. Work of the third and fourth authors was supported by the National Science Foundation, grant DMS-0901642.

E-mail addresses: Alexandru.Aleman@math.lu.se (A. Aleman), karl-mikael.perfekt@math.ntnu.no (K.-M. Perfekt), richter@math.utk.edu (S. Richter), sundberg@math.utk.edu (C. Sundberg).

http://dx.doi.org/10.1016/j.jfa.2015.01.012
0022-1236/

are defined by linear relations of the graph transformations for $\mathcal{M}$.

1. Introduction

Let $d \geq 1$, $\Omega \subseteq \mathbb{C}^d$ be an open, connected, and nonempty set, and let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space. If $\varphi \in \mathrm{Hol}(\Omega)$ such that $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$, then φ is called a multiplier and $M_\varphi f = \varphi f$ defines a bounded linear operator on $\mathcal{H}$. We use $\mathcal{M}(\mathcal{H})$ to denote the multiplier algebra of $\mathcal{H}$, $\mathcal{M}(\mathcal{H}) = \{M_\varphi \in \mathcal{B}(\mathcal{H}) : \varphi \text{ is a multiplier}\}$.

A subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a transitive algebra if it contains the identity operator and if it has no nontrivial common invariant subspaces. It is a longstanding open question (due to Kadison), called the transitive algebra problem, to decide whether every transitive algebra is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology. If that were the case, then, as is well-known, it would easily follow that every $T \in \mathcal{B}(\mathcal{H})$ which is not a scalar multiple of the identity has a nontrivial hyperinvariant subspace (see e.g. [24]). Recall that a subspace $\mathcal{M}$ is called hyperinvariant for an operator A, if it is invariant for every bounded operator that commutes with A.

Arveson was the first to systematically study the transitive algebra problem. We say that an operator A (respectively an algebra $\mathcal{A}$) has the transitive algebra property, if every transitive algebra that contains A (respectively $\mathcal{A}$) is strongly dense in $\mathcal{B}(\mathcal{H})$. Arveson showed that any maximal abelian self-adjoint subalgebra and the unilateral shift have the transitive algebra property. We refer the reader to [24] for further early results on the transitive algebra problem.

Arveson's approach requires a detailed knowledge of the invariant subspace structure of the operator or the algebra that is to be shown to have the transitive algebra property. Thus based on information about the invariant subspaces of the Dirichlet space Richter was able to use Arveson's approach to establish that the Dirichlet shift has the transitive algebra property, [26]. Then more generally Chong, Guo and Wang, [11], followed a similar strategy to show among other things that $\mathcal{M}(\mathcal{H})$ has the transitive algebra property, whenever $\mathcal{H}$ has a complete Nevanlinna–Pick kernel, i.e. if the reproducing kernel $k_\lambda(z)$ for $\mathcal{H}$ is of the form $k_\lambda(z) = \frac{\overline{f(\lambda)}f(z)}{1-u_\lambda(z)}$, where f is an analytic function and $u_\lambda(z)$ is positive definite and sesquianalytic. This result covers both the unilateral shift and the Dirichlet shift, and without going into further detail we should say that the Chong–Guo–Wang result also covers higher finite multiplicities as well as restrictions to invariant subspaces.

The current paper was motivated by the desire to decide which other multiplier algebras have the transitive algebra property. Although we did not obtain any specific answers, our investigations lead us to consider some interesting questions related to the invariant subspace structure of $\mathcal{M}(\mathcal{H})$. For additional recent work on questions about transitive algebras we refer the reader to [9].