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Continuous Renormalization Group Analysis of Spectral Problems in Quantum Field Theory

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A B S T R A C T

The isospectral renormalization group is a powerful method to analyze the spectrum of operators in quantum field theory. It was introduced in 1995 (see [2,4]) and since then it has been used to prove several results for non-relativistic quantum electrodynamics. After the introduction of the method there have been many works in which extensions, simplifications or clarifications are presented (see [7,11,13]). In this paper we present a new approach in which we construct a flow of operators parametrized by a continuous variable in the positive real axis. While this is in contrast to the discrete iteration used before, this is more in spirit of the original formulation of the renormalization group introduced in theoretical physics in 1974 [22]. The renormalization flow that we construct can be expressed in a simple way: it can be viewed as a single application of the Feshbach–Schur map with a clever selection of the spectral parameter. Another advantage of the method is that there exists a flow function for which the renormalization group that we present is the orbit under this flow of an initial Hamiltonian. This opens

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the possibility to study the problem using different techniques coming from the theory of evolution equations.

1. Introduction

1.1. Historical Context and Description of the Problem

The processes of emission and absorption of photons by atoms can be rigorously understood in the low-energy limit, if we neglect the creation and annihilation of electrons. The corresponding theory is frequently referred to as *nonrelativistic quantum electrodynamics (NR QED)*. The description of matter and light from the mathematical point of view relies on the study of eigenvalues of operators, which are immersed in the continuum. The study of eigenvalues immersed in the continuum requires sophisticated constructions that do not fall into the realm of regular perturbation theory used to analyze isolated eigenvalues (see [19]). There are two methods that have been applied to investigate these questions. The first one, introduced in [2–4], is the spectral renormalization group. Inspired by a construction that Feshbach used in [9], the Feshbach–Schur projection map is defined and developed [2–4]. The method is based on a transformation that allows for a localization of the regions of the spectrum that we are interested in. The second method, introduced in [17], produces a sequence of isolated eigenvalues that converges to the desired eigenvalue, which is immersed in the continuum, by including ever more momentum shells into the dynamics.

The spectral renormalization group has been extensively used to analyze spectral problems in nonrelativistic quantum electrodynamics. In numerous works this method has been used to prove basic properties of the spectrum of Hamiltonian operators for different models (see for example [1–8] and [10–16]). Although it is a powerful tool to analyze the spectrum of operators, it yet has the disadvantage of being conceptually and technically complicated. Further developments of the original method have been presented in [7,11,13]. In these works, new techniques and methods are presented that simplify the computations and clarify the concepts of the original procedures introduced in [2] and [4]. In this paper we present a new approach to the renormalization group described in the following section. Interestingly, this new approach uses the spatial length scale as a flow parameter and is thus closer to the original renormalization group introduced by Kogut and Wilson in [22] and subsequently improved by Polchinski [18] and later Wieczerkowski [21] and Salmhofer [20].

1.2. Short Description of the Main Results

In this section we describe our main results. We present a short description of the method and the most important theorems without giving precise definitions of the operators and spaces that we use. The definitions are deferred to later sections.