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# Compact composition operators with symbol a universal covering map

Matthew M. Jones

*Mathematics Department, Middlesex University, Hendon, London, UK*

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In this paper we study composition operators, $C_\phi$, acting on the Hardy spaces that have symbol, $\phi$, a universal covering map of the disk onto a finitely connected domain of the form $\mathcal{D}_0 \backslash \{p_1, \dots, p_n\}$, where $\mathcal{D}_0$ is simply connected and $p_i$, $i = 1, \dots, n$, are distinct points in the interior of $\mathcal{D}_0$. We consider, in particular, conditions that determine compactness of such operators and demonstrate a link with the Poincare series of the uniformizing Fuchsian group. We show that $C_\phi$ is compact if, and only if $\phi$ does not have a finite angular derivative at any point of the unit circle, thereby extending the result for univalent and finitely multivalent $\phi$.



## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disk in the complex plane, then the Hardy space $H^p$, $1 \le p < \infty$, is defined to be the Banach space of functions holomorphic in $\mathbb{D}$ with norm

$$\|f\|_p^p = \lim_{r \to 1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p d\theta < \infty.$$

*E-mail address:* m.m.jones@mdx.ac.uk.

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The limit here is guaranteed by the fact that the integral mean is increasing in $r$. The standard text for the theory of Hardy spaces is [6].

Given a holomorphic map $\phi\colon \mathbb{D} \to \mathbb{D}$ we define the composition operator

$$C_\phi\colon f \to f \circ \phi.$$

The study of composition operators acting on function spaces has received much attention over the last four decades. The central theme of this work is to understand how operator theoretic properties of composition operators are related to geometric or analytic properties of their inducing functions. Of central importance in this area is a result of Shapiro, [10], which describes the essential norm of a composition operator in terms of the Nevanlinna counting function of its inducing holomorphic map. The Nevanlinna counting function is known explicitly in a number of situations, for example for inner functions, univalent functions and finitely multivalent functions.

In this paper we study composition operators with symbol a universal covering map of the unit disk onto a finitely connected domain, in this case the Nevanlinna counting function can be estimated precisely by properties of the underlying Fuchsian group. We will provide all the preliminary definitions in Section 2.

We consider throughout this article domains of the form

$$\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}, \quad n \geq 1 \tag{1}$$

where $\mathcal{D}_0$ is a simply connected domain contained in $\mathbb{D}$ and $p_1, \ldots, p_n$ are distinct, isolated points in the interior of $\mathcal{D}_0$. We will study composition operators whose symbol $\phi$ is the universal covering map of $\mathbb{D}$ onto $\mathcal{D}$.

For a Fuchsian group $\Gamma$ we define the limit set $\Lambda(\Gamma)$ to be the set of accumulation points of orbits of points in $\mathbb{D}$ by functions in $\Gamma$. The Poincare series for $\Gamma$ of order $s$ is

$$\rho_\Gamma(z, w; s) = \sum_{g \in \Gamma} \exp -s d_{\mathbb{D}}\big(z, g(w)\big) \tag{2}$$

where $d_{\mathbb{D}}(z, w)$ is the hyperbolic distance from $z$ to $w$ in $\mathbb{D}$.

It is known that there is a critical exponent, $\delta(\Gamma)$ such that the Poincare series converges for all $s < \delta(\Gamma)$ but diverges for all $s > \delta(\Gamma)$. For finitely generated Fuchsian groups

$$\delta(\Gamma) = \dim\big(\Lambda(\Gamma)\big),$$

the Hausdorff dimension of the limit set of $\Gamma$.

A simple calculation shows that if $\Gamma$ is elementary and generated by a parabolic element then

$$\delta(\Gamma) = 1/2.$$