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The Higson–Roe exact sequence and ℓ^2 eta invariants

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ABSTRACT

The goal of this paper is to solve the problem of existence of an ℓ^2 relative eta morphism on the Higson–Roe structure group. Using the Cheeger–Gromov ℓ^2 eta invariant, we construct a group morphism from the Higson–Roe maximal structure group constructed in [35] to the reals. When we apply this morphism to the structure class associated with the spin Dirac operator for a metric of positive scalar curvature, we get the spin ℓ^2 rho invariant. When we apply this morphism to the structure class associated with an oriented homotopy equivalence, we get the difference of the ℓ^2 rho invariants of the corresponding signature operators. We thus get new proofs for the classical ℓ^2 rigidity theorems of Keswani obtained in [41].

Contents

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1. Introduction

The eta invariant of elliptic operators first appeared in [3] as a boundary correction term appearing in the calculation of the index of a Fredholm operator associated with a global boundary value problem on even dimensional manifolds with boundary. The eta invariant $\eta(D)$ is a measure of asymmetry of the spectrum of the operator D and turns out to be well-defined for any elliptic self-adjoint differential operators D on a closed odd dimensional manifold M. This is a sensitive invariant, but there is a relative version which is more stable and often has interesting topological properties. More precisely, given two group morphisms $\sigma_1, \sigma_2 : \pi_1(M) \to U(N)$ and the associated flat bundles E_{σ_i}, we may form the twisted elliptic differential operators $D \otimes E_{\sigma_i}$ and the relative eta invariant is by definition [4,5]

$$\rho_{\sigma_1,\sigma_2}(D) := \eta(D \otimes E_{\sigma_1}) - \eta(D \otimes E_{\sigma_2}).$$

If D is for instance the signature operator on M then it was proved by Atiyah, Patodi and Singer that $\rho_{\sigma_1,\sigma_2}(D)$ is a differential invariant of M. This property had important consequences, as when $\pi_1(M)$ has torsion this invariant is not a homotopy invariant, see for instance [20,21,45,49,57]. Notice that the relative index is zero thanks to the Atiyah–Singer index formula and the relative eta invariant can thus be seen as a refined *secondary* invariant, in fact some transgression of the index [23,44]. In general, when reduced modulo $\mathbb{Z}$, this invariant becomes more computable and inherits topological properties, there is indeed a topological index formula in $\mathbb{R}/\mathbb{Z}$ which expresses it in terms of characteristic classes [5].

In [22], Cheeger and Gromov extended the APS eta invariant and introduced an ℓ^2 version of the eta invariant exactly as Atiyah introduced an ℓ^2 version of the index. More precisely, given a Galois Γ-covering $\widetilde{M} \to M$ and a Γ-invariant generalized Dirac operator $\widetilde{D}$ over $\widetilde{M}$, the Cheeger–Gromov eta invariant is defined by the absolutely convergent integral [22]: