Journal of Functional Analysis 266 (2014) 3435–3454

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Isoperimetric inequality for radial probability measures on Euclidean spaces

Asuka Takatsu [1]

Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan

ARTICLE INFO

Article history:
Received 15 January 2013
Accepted 24 January 2014
Available online 4 February 2014
Communicated by Cédric Villani

Keywords:
Poincaré limit
Isoperimetric inequality

ABSTRACT

We generalize the Poincaré limit which asserts that the n-dimensional Gaussian measure is approximated by the projections of the uniform probability measure on the Euclidean sphere of appropriate radius to the first n-coordinates as the dimension diverges to infinity. The generalization is done by replacing the projections with certain maps. Using this generalization, we derive a Gaussian isoperimetric inequality for an absolutely continuous probability measure on Euclidean spaces with respect to the Lebesgue measure, whose density is a radial function.

1. Introduction

The isoperimetric profile of a Borel probability measure μ on $\mathbb{R}^n$ describes a relation between the volume $\mu[A]$ and the *boundary measure* $\mu^+[A] := \underline{\lim}_{\varepsilon\downarrow 0}(\mu[A^\varepsilon] - \mu[A])/\varepsilon$ of $A \subset \mathbb{R}^n$, where $A^\varepsilon := \{x \in \mathbb{R}^n \mid \inf_{a\in A} |x - a| < \varepsilon\}$ denotes the ε-neighborhood of A with respect to the standard Euclidean norm $|\cdot|$. Throughout this note, any subset of $\mathbb{R}^n$ is assumed to be Borel. Precisely, the *isoperimetric profile* $I[\mu]$ of μ is a function on $[0, 1]$ defined by

$$I[\mu](a) := \inf\bigl\{\mu^+[A] \bigm| A \subset \mathbb{R}^n \text{ with } \mu[A] = a\bigr\}.$$

E-mail address: takatsu@math.nagoya-u.ac.jp.

[1] Supported in part by the Grant-in-Aid for Young Scientists (B) 24740042.

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2014.01.014

Let A_n denote the boundary measure of the unit ball in $\mathbb{R}^n$ with respect to the Lebesgue measure. For a measurable, nonnegative function f on $(0,\infty)$ satisfying

$$M_n^f := A_n \int_0^\infty f(r) r^{n-1}\, dr < \infty,$$

define the n-dimensional *radial probability measure* μ_n^f *with density* f as the absolutely continuous probability measure on $\mathbb{R}^n$ with density

$$\frac{d\mu_n^f}{dx}(x) = \frac{1}{M_n^f} f\bigl(|x|\bigr)$$

with respect to the Lebesgue measure. For example, the n-dimensional Gaussian measure γ_n is the radial probability measure with density $g(r) := \exp(-r^2/2)$, and its isoperimetric profile was provided by Borell [3] and Sudakov and Tsirel'son [6] independently of the form

$$I[\gamma_n](a) = I[\gamma_1](a) = G'\bigl(G^{-1}(a)\bigr), \qquad G(r) := \int_{-\infty}^{r} (2\pi)^{-1/2} g(s)\, ds = \gamma_1\bigl[(-\infty, r]\bigr],$$

where the infimum in the definition of $I[\gamma_n](a)$ is attained by the hyperplane of the form

$$H_a := \bigl\{x \in \mathbb{R}^n \bigm| x_1 < G^{-1}(a)\bigr\}.$$

The proof relies on the approximation procedure, so-called Poincaré limit: let S_N be the $(N-1)$-dimensional Euclidean sphere of radius $N^{1/2}$ and v_N be the uniform probability measure on S_N. We consider the orthogonal projection from $\mathbb{R}^N$ to the first n-coordinates, and denote by $P_{n,N}$ the restriction of it on S_N. Then γ_n is obtained as the weak limit of $P_{n,N\sharp} v_N$ as $N \to \infty$, where $P_{n,N\sharp} v_N$ denotes the push-forward measure of v_N by $P_{n,N}$, namely $P_{n,N\sharp} v_N[A] = v_N[(P_{n,N})^{-1}(A)]$ for any $A \subset \mathbb{R}^n$.

The aim of this note is to derive a Gaussian isoperimetric inequality for μ_n^f, that is, estimate $I[\mu_n^f]$ below by $I[\gamma_1]$. To do this, let us generalize the Poincaré limit by replacing $P_{n,N}$ with $P_{n,N}^\rho := s_n^\rho \circ P_{n,N}$, where s_n^ρ is the map on $\mathbb{R}^n$ defined as

$$s_n^\rho(x) := \begin{cases} \rho(|x|)x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

for a function ρ on $(0,\infty)$ satisfying the following condition.

(C) ρ is C^1, positive in $(0,\infty)$ and s_1^ρ is strictly increasing.

Theorem 1.1. *For a function ρ satisfying* (C)*, let σ be the inverse function of s_1^ρ. For any $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_{n,N}^\rho(x) := d(P_{n,N\sharp}^\rho v_N)(x)/dx\}_{N\in\mathbb{N}}$ converges to*