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# On differential operators associated with Markov operators

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ABSTRACT

In this paper we introduce and study a new class of elliptic second-order differential operators on a convex compact subset $K$ of $\mathbf{R}^d$, $d \geqslant 1$, which are associated with a Markov operator $T$ on $\mathscr{C}(K)$ and which degenerate on a suitable subset of $K$ containing its extreme points. Among other things, we show that the closures of these operators generate Markov semigroups. Moreover, we prove that these semigroups can be approximated by means of iterates of suitable positive linear operators, which are referred to as the Bernstein–Schnabl operators associated with $T$. As a consequence we show that the semigroups preserve polynomials of a given degree as well as Hölder continuity, which gives rise to some spatial regularity properties of the solutions of the relevant evolution equations.


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## 1. Introduction

This paper is mainly concerned with the study of a new class of second-order differential operators on compact subsets of $\mathbf{R}^d$, $d \geqslant 1$, which can be associated with a Markov operator.

More precisely, given a convex compact subset $K$ of $\mathbf{R}^d$ with non-empty interior and a Markov operator $T$ on $\mathscr{C}(K)$ (i.e., a positive linear operator $T$ on $\mathscr{C}(K)$ such that $T(\mathbf{1}) = \mathbf{1}$), we shall consider and study the following elliptic second-order differential operator $W_T$, defined by setting, for every $u \in \mathscr{C}^2(K)$,

$$W_T(u) := \frac{1}{2} \sum_{i,j=1}^{d} \alpha_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1}$$

where, for each $i, j = 1, \ldots, d$, $\alpha_{ij} := T(pr_i pr_j) - pr_i pr_j$ and $pr_i$ stands for the $i$-th coordinate function.

One of the difficulties in studying operators (1) lies in the fact that the boundary $\partial K$ of $K$ is generally non-smooth, due to the presence of possible sides and corners; moreover, since we assume that $T$ preserves the coordinate functions, the operator $W_T$ degenerates on a subset of $K$ containing the set $\partial_e K$ of all the extreme points of $K$.

On the other hand, operators (1) are of concern in the study of several diffusion problems arising in biology, financial mathematics and other fields, so that they seem to be worthy of a comprehensive and thorough study.

In this paper we are mainly interested in proving that, under suitable assumptions on $T$, the operator $(W_T, \mathscr{C}^2(K))$ is closable and its closure is the generator of a Markov semigroup $(T(t))_{t \geqslant 0}$ on $\mathscr{C}(K)$.

Our approach is based on a Trotter-type result due to Schnabl [25]; in fact, in order to prove our generation result, we consider a suitable sequence $(B_n)_{n \geqslant 1}$ of positive linear operators, the so-called *Bernstein–Schnabl operators* associated with $T$, which are strictly connected with operator (1) via an asymptotic formula, and then, by means of them, we obtain the generation result by also showing that the generated semigroup $(T(t))_{t \geqslant 0}$ can be represented in terms of suitable iterates of the $B_n$'s. As a consequence we show that the semigroups preserve polynomials of a given degree as well as Hölder continuity, which, in turn, allows to highlight some spatial regularity properties of the solutions of the relevant evolution equations.

The sequence of Bernstein–Schnabl operators is an approximation process in $\mathscr{C}(K)$, so that, from the point of view of Approximation Theory, it seems to have an interest on its own. For this reason, even if in this work we investigate several properties of $(B_n)_{n \geqslant 1}$, we shall undertake a deeper and more accurate study of them in a forthcoming monograph [14]. In the same monograph we shall also deepen the investigations of other