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On the Hardy constant of non-convex planar domains: The case of the quadrilateral

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Abstract

The Hardy constant of a simply connected domain $\Omega \subset \mathbb{R}^2$ is the best constant for the inequality

$$\int_\Omega |\nabla u|^2\,dx \geqslant c \int_\Omega \frac{u^2}{\mathrm{dist}(x,\partial\Omega)^2}\,dx, \quad u \in C_c^\infty(\Omega).$$

After the work of Ancona where the universal lower bound $1/16$ was obtained, there has been a substantial interest on computing or estimating the Hardy constant of planar domains. In this work we determine the Hardy constant of an arbitrary quadrilateral in the plane. In particular we show that the Hardy constant is the same as that of a certain infinite sectorial region which has been studied by E.B. Davies.

Keywords: Hardy inequality; Hardy constant; Distance function

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1. Introduction

In the 1920's Hardy established the following inequality [12]:

$$\int_0^\infty u'(t)^2\,dt \geqslant \frac{1}{4}\int_0^\infty \frac{u^2}{t^2}\,dt, \qquad \text{for all } u \in C_c^\infty(0,\infty). \tag{1}$$

The constant $1/4$ is the best possible, and equality is not attained for any non-zero function in the appropriate Sobolev space.

Inequality (1) immediately implies the following inequality on $\mathbb{R}_+^N = \mathbb{R}^{N-1} \times (0,+\infty)$:

$$\int_{\mathbb{R}_+^N} |\nabla u|^2\,dx \geqslant \frac{1}{4}\int_{\mathbb{R}_+^N} \frac{u^2}{x_N^2}\,dx, \qquad \text{for all } u \in C_c^\infty\big(\mathbb{R}_+^N\big), \tag{2}$$

where again the constant $1/4$ is the best possible. The analogue of (2) for a domain $\Omega \subset \mathbb{R}^N$ is

$$\int_\Omega |\nabla u|^2\,dx \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,dx, \qquad \text{for all } u \in C_c^\infty(\Omega), \tag{3}$$

where $d = d(x) = \operatorname{dist}(x,\partial\Omega)$. However, (3) is not true without geometric assumptions on Ω. The typical assumption made for the validity of (3) is that Ω is convex [10]. A weaker geometric assumption introduced in [7] is that Ω is weakly mean convex, that is

$$-\Delta d(x) \geqslant 0, \qquad \text{in } \Omega, \tag{4}$$

where Δd is to be understood in the distributional sense. Condition (4) is equivalent to convexity when $N = 2$ but strictly weaker than convexity when $N \geqslant 3$ [4].

In the last years there has been a lot of activity on Hardy inequality and improvements of it under the convexity or weak mean convexity assumption on Ω; see [8,7,13,11]. If no geometric assumptions are imposed on Ω, then one can still obtain inequalities of similar type. If for example Ω is bounded with C^2 boundary then one can still have inequality (3) for all $u \in C_c^\infty(\Omega_\epsilon)$ where $\Omega_\epsilon = \{x \in \Omega\colon\ d(x) < \epsilon\}$, provided $\epsilon > 0$ is small enough [11]. In the same spirit, under the same assumptions on Ω it was proved in [8] that there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega |\nabla u|^2\,dx + \lambda\int_\Omega u^2\,dx \geqslant \frac{1}{4}\int_\Omega \frac{u^2}{d^2}\,dx, \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{5}$$

More generally, it is well known that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ there exists $c > 0$ such that

$$\int_\Omega |\nabla u|^2\,dx \geqslant c\int_\Omega \frac{u^2}{d^2}\,dx, \qquad \text{for all } u \in C_c^\infty(\Omega). \tag{6}$$

Following [9] we call the best constant c of inequality (6) the Hardy constant of the domain Ω.