J. Funct. Anal. 266 (2014) 3797–3814

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Basic sequences and spaceability in ℓ_p spaces [☆]

Daniel Cariello [a,b], Juan B. Seoane-Sepúlveda [a,c,*]

[a] Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Plaza de Ciencias 3, Universidad Complutense de Madrid, Madrid, 28040, Spain

[b] Faculdade de Matemática, Universidade Federal de Uberlândia, 38.400-902, Uberlândia, Brazil

[c] Instituto de Ciencias Matemáticas – ICMAT, calle Nicolás Cabrera 13–15, Madrid, 28049, Spain

ARTICLE INFO

Article history:
Received 15 August 2013
Accepted 12 December 2013
Available online 27 December 2013
Communicated by G. Schechtman

Keywords:
Lineability
Spaceability
Algebrability
Basic sequence
Complemented subspace
ℓ_p spaces

ABSTRACT

Let X be a sequence space and denote by $Z(X)$ the subset of X formed by sequences having only a finite number of zero coordinates. We study algebraic properties of $Z(X)$ and show (among other results) that (for $p \in [1, \infty]$) $Z(\ell_p)$ does not contain infinite dimensional closed subspaces. This solves an open question originally posed by R.M. Aron and V.I. Gurariy in 2003 on the linear structure of $Z(\ell_\infty)$. In addition to this, we also give a thorough analysis of the existing algebraic structures within the sets $Z(X)$ and $X \setminus Z(X)$ and their algebraic genericities.

1. Introduction and preliminaries

During a *Non-linear Analysis Seminar* at Kent State University (Kent, Ohio, USA) in 2003, Richard M. Aron and Vladimir I. Gurariy posed the following question:

Question 1.1 *(R. Aron and V. Gurariy, 2003).* Is there an infinite dimensional closed subspace of ℓ_∞ every nonzero element of which has only a finite number of zero coordinates?

☆ D. Cariello was supported by CNPq-Brazil Grant 245277/2012-9. J.B. Seoane-Sepúlveda was supported by CNPq Grant 401735/2013-3 (PVE – Linha 2).

* Corresponding author.
E-mail addresses: dcariello@famat.ufu.br (D. Cariello), jseoane@mat.ucm.es (J.B. Seoane-Sepúlveda).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.12.011

Using modern terminology (originally coined by V. Gurariy himself), a subset M of a topological vector space X is called *lineable* (resp. *spaceable*) in X if there exists an infinite dimensional linear space (resp. an infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$ (see [1,9,13,16]). V. Gurariy also coined the notion of *algebrability* (introduced in [2]) meaning that, given a Banach algebra $\mathcal{A}$ and a subset $\mathcal{B} \subset \mathcal{A}$, it is said that $\mathcal{B}$ is *algebrable* if there exists a subalgebra $\mathcal{C}$ of $\mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite.

Throughout this paper, and if X denotes a sequence space, we shall denote by $Z(X)$ the subset of X formed by sequences having only a finite number of zero coordinates. Therefore, the above question can be stated in terms of lineability and spaceability:

Is $Z(\ell_\infty)$ spaceable in ℓ_∞?

Lately, these concepts of lineability and spaceability have proven to be quite fruitful and have attracted the interest of many mathematicians, among whom we have R. Aron, F. Bastin, L. Bernal-González, P. Enflo, G. Godefroy, V. Fonf, V. Gurariy, V. Kadets, or E. Teixeira (see, e.g. [3–6,9–11,13,20]). Question 1.1 has also appeared in several recent works (see, e.g., [9,13,14,19]) and, for the last decade, there have been several attempts to partially answer it, although nothing conclusive in relation to the original problem has been obtained so far.

This paper is arranged as follows. Section 2 shall focus on the algebrability (and, thus, lineability) of the set $Z(X)$ for $X \in \{c_0, \ell_p\}$, $p \in [1, \infty]$. Sections 3 and 4 will show that spaceability of $Z(X)$ is actually not possible for any of the previous Banach spaces whereas, in Section 5, we shall show that $V \setminus Z(V)$ is, actually, spaceable (and algebrable) for every infinite dimensional closed subspace (subalgebra) V of X (for $X \in \{c_0, \ell_p\}$, $p \in [1, \infty]$).

There are not many examples of (nontrivial) sets that are lineable and not spaceable. One of the first ones in this direction, is due to B. Levine and D. Milman (1940, [18]) who showed that the subset of $\mathcal{C}[0,1]$ of all functions of bounded variation is not spaceable (it is obviously lineable, since it is an infinite dimensional linear space itself). A more recent one is due to V. Gurariy (1966, [15]), who showed that the set of everywhere differentiable functions on $[0,1]$ (which is also an infinite dimensional linear space) is not spaceable in $\mathcal{C}([0,1])$. However, L. Bernal-González [7] showed that $\mathcal{C}^\infty(]0,1[)$ is, actually, spaceable in $\mathcal{C}(]0,1[)$.

Here, we shall provide (among other results) the definitive answer to Question 1.1. Namely, if X stands for c_0, or ℓ_p, with $p \in [1, \infty]$, we prove the following:

(i) $Z(X)$ is maximal algebrable and maximal lineable [8], that is, $Z(X) \cup \{0\}$ contains an algebra with a system of generators of cardinality $\dim(X)$ and a linear subspace of dimension $\dim(X)$ (Proposition 2.1).

(ii) $Z(X)$ is not spaceable, that is, every closed subspace of $Z(X) \cup \{0\}$ must have finite dimension (Corollaries 3.4 and 4.8).