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Journal of Functional Analysis 266 (2014) 3894–3905

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Optimal average approximations for functions mapping in quasi-Banach spaces

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Received 31 August 2013; accepted 15 November 2013

Available online 27 November 2013

Communicated by G. Schechtman

Abstract

In 1994, M.M. Popov [6] showed that the fundamental theorem of calculus fails, in general, for functions mapping from a compact interval of the real line into the ℓ_p-spaces for $0 < p < 1$, and the question arose whether such a significant result might hold in some other non-Banach spaces. In this article we completely settle the problem by proving that the fundamental theorem of calculus breaks down in the context of *any* non-locally convex quasi-Banach space. Our approach introduces the tool of Riemann-integral averages of continuous functions, and uses it to bring out to light the differences in behavior of their approximates in the lack of local convexity. As a by-product of our work we solve a problem raised in [1] on the different types of spaces of differentiable functions with values on a quasi-Banach space.

Keywords: Riemann integral; Optimal approximation; Quasi-Banach space; Fundamental theorem of calculus

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[1] F. Albiac acknowledges the support of the Spanish Ministry for Economy and Competitivity Grant *Operators, lattices, and structure of Banach spaces*, reference number MTM 2012-31286, and the Spanish Ministry for Science and Research Grant *Structure and complexity in Banach spaces II*, reference number MTM2010-20190-C02-02.

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.11.015

1. Introduction and background

Continuous maps from a compact interval of the real line into a Banach space X are Riemann-integrable. For each $f : [a, b] \to X$, the optimal behavior of the averages

$$\frac{1}{t-s}\int_s^t f(u)\,du, \quad a \leqslant s < t \leqslant b,$$

in approximating f locally in norm is substantiated by the fact that, thanks to the convexity of the space, for every point $c \in [a, b]$ we have

$$\left\| \frac{1}{t-s}\int_s^t f(u)\,du - f(c) \right\| = \left\| \int_s^t \frac{f(u)-f(c)}{t-s}\,du \right\| \leqslant \max_{u\in[s,t]} \left\| f(u) - f(c) \right\|,$$

so that

$$\lim_{(s,t)\to(c,c)} \frac{1}{t-s}\int_s^t f(u)\,du = f(c).$$

In other words, the formula

$$F(s,t) = \operatorname{Ave}[f](s,t) = \begin{cases} \frac{1}{t-s}\int_s^t f(u)\,du, & \text{if } a \leqslant s < t \leqslant b, \\ f(c), & \text{if } a \leqslant s = t = c \leqslant b, \\ \frac{1}{s-t}\int_t^s f(u)\,du, & \text{if } a \leqslant t < s \leqslant b, \end{cases} \tag{1.1}$$

defines a *jointly continuous* function from $[a, b] \times [a, b]$ into X. In particular, $\operatorname{Ave}[f]$ is *bounded*, i.e.,

$$\sup_{a\leqslant s,t\leqslant b} \left\| \operatorname{Ave}[f](s,t) \right\| < \infty,$$

and *separately continuous*, i.e., for fixed s_0 and t_0 in $[a, b]$ we have

$$\lim_{s\to s_0} \operatorname{Ave}[f](s,t_0) = \lim_{t\to t_0} \operatorname{Ave}[f](s_0,t) = \operatorname{Ave}[f](s_0,t_0).$$

The study of averages of continuous functions mapping into quasi-Banach spaces faces obstructions from the very beginning. Indeed, if X is non-locally convex, by an old result of Mazur and Orlicz [5] there exist continuous X-valued functions failing to be Riemann-integrable. Thus, to extend the problem we suppose that $f : [a, b] \to X$ is continuous and integrable, and wonder whether, with this extra assumption, the average function defined in (1.1) will retain the optimality it enjoys for Banach spaces. The answer to this question is negative, as the authors showed in [1].