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# Zappa–Szép products of semigroups and their $C^*$-algebras ☆

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A B S T R A C T

Zappa–Szép products of semigroups provide a rich class of examples of semigroups that include the self-similar group actions of Nekrashevych. We use Li's construction of semigroup $C^*$-algebras to associate a $C^*$-algebra to Zappa–Szép products and give an explicit presentation of the algebra. We then define a quotient $C^*$-algebra that generalises the Cuntz–Pimsner algebras for self-similar actions. We indicate how known examples, previously viewed as distinct classes, fit into our unifying framework. We specifically discuss the Baumslag–Solitar groups, the binary adding machine, the semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$, and the $ax+b$-semigroup $\mathbb{Z} \rtimes \mathbb{Z}^\times$.


## 1. Introduction

Examples are crucial to progress in $C^*$-algebras. Operator-algebraists are therefore enthusiastic to have ways of generating and analysing rich classes of examples. Semigroups feature in a number of families of interesting examples. In this article we describe a new class of semigroup $C^*$-algebras.

The theory of $C^*$-algebras associated to semigroups can be traced back to Coburn's Theorem [3], which says that any two $C^*$-algebras generated by a non-unitary isometry are isomorphic. There have been a number of generalisations of Coburn's Theorem,

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including Douglas's work [10] on positive cones of ordered subgroups of $\mathbb{R}$, and Murphy's work [20] on positive cones in ordered abelian groups. A major generalisation was developed by Nica [24] through his introduction of quasi-lattice ordered groups $(G, P)$.

A quasi-lattice ordered group $(G, P)$ consists of a partially-ordered group $G$ and a positive cone $P$ in $G$. Nica identified a class of covariant isometric representations of $P$, and introduced the $C^*$-algebra $C^*(G, P)$ universal for such representations. Quasi-lattice ordered groups are rigid enough in their structure to produce a tractable class of $C^*$-algebras $C^*(G, P)$, and yet they include a wide range of interesting semigroups as examples. Indeed, quasi-lattice ordered groups are still providing a rich source of interesting $C^*$-algebras, as evidenced by the recent work on the $C^*$-algebras associated to $\mathbb{N} \rtimes \mathbb{N}^\times$ [14], and the Baumslag–Solitar groups [27].

A broad generalisation of Nica's $C^*$-algebras associated to quasi-lattice ordered groups has recently been introduced by Li [18]. He associates a number of $C^*$-algebras to discrete left cancellative semigroups. This generality is possible because of the importance of the right ideal structure of the semigroup; the full $C^*$-algebra $C^*(P)$ is generated by an isometric representation of $P$ and a family of projections associated to right ideals in $P$ satisfying a set of relations. As well as quasi-lattice ordered groups, Li's construction caters for the $ax+b$-semigroups over the rings of algebraic integers in number fields (see also [6]). We will examine the $ax + b$-semigroup over $\mathbb{Z}$, which is the ring of algebraic integers in $\mathbb{Q}$.

A seemingly unrelated class of $C^*$-algebras has recently been discovered by Nekrashevych [21,23], namely those associated with self-similar group actions. The first example of a self-similar action was given by Grigorchuk [11]. As an infinite finitely-generated torsion group with intermediate growth, Grigorchuk's example solved a number of open problems, see [22, p. 14]. Since then a large number of interesting group actions have been shown to be self-similar and we refer the reader to Nekrashevych's book [22] for further details.

A self-similar action $(G, X)$ consists of a group $G$ with a faithful action on the set $X^*$ of finite words on a finite set $X$; the action is self-similar in the sense that for each $g \in G$ and $x \in X$ there exists a unique $g|_x \in G$ such that $g \cdot (xw) = (g \cdot x)(g|_x \cdot w)$ for all $w \in X^*$. Nekrashevych associated a Cuntz–Pimsner $C^*$-algebra to a self-similar action $(G, X)$ via generators and relations. The algebra is generated by a unitary representation of $G$ and a collection of isometries associated to $X$, with commutation relations modelled on the self-similarity relations. The Cuntz–Pimsner algebra contains copies of the full group $C^*$-algebra and the Cuntz algebra $\mathcal{O}_{|X|}$. Since then, a universal Toeplitz–Cuntz–Pimsner algebra has been constructed that contains a generalised version of Nekrashevych's algebras as a quotient [15]. The self-similar commutation relations provide for an extremely simple generating set in both cases, and make the algebras particularly tractable. These commutation relations have been the inspiration for the results in this paper.

We identify a class of $C^*$-algebras that includes both those associated to quasi-lattice ordered groups and those associated to self-similar actions. We do this using a construction that was developed by G. Zappa in [31] and J. Szép in [28–30]. Given two groups,