Journal of Functional Analysis 267 (2014) 4635–4666

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Concerning L^p resolvent estimates for simply connected manifolds of constant curvature ☆

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ARTICLE INFO

Article history:
Received 7 July 2014
Accepted 19 August 2014
Available online 29 August 2014
Communicated by Daniel W. Stroock

Keywords:
Resolvent
Eigenfunctions
Constant curvature

ABSTRACT

We prove families of uniform (L^r, L^s) resolvent estimates for simply connected manifolds of constant curvature (negative or positive) that imply the earlier ones for Euclidean space of Kenig, Ruiz and the second author [7]. In the case of the sphere we take advantage of the fact that the half-wave group of the natural shifted Laplacian is periodic. In the case of hyperbolic space, the key ingredient is a natural variant of the Stein–Tomas restriction theorem.

1. Introduction and main results

In a paper of Kenig, Ruiz and the second author [7], it was shown that for each $n \geq 3$ if $1 < r < s < \infty$ are Lebesgue exponents satisfying

$$n\left(\frac{1}{r} - \frac{1}{s}\right) = 2 \quad \text{and} \quad \min\left(\left|\frac{1}{r} - \frac{1}{2}\right|, \left|\frac{1}{s} - \frac{1}{2}\right|\right) > \frac{1}{2n}, \tag{1.1}$$

☆ The first author was visiting Johns Hopkins University while this research was carried out, supported by the China Scholarship Council (201306160006). The second was supported in part by the NSF, DMS-1361476.

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http://dx.doi.org/10.1016/j.jfa.2014.08.016
0022-1236/

then there is a uniform constant $C_{r,s} < \infty$ so that

$$\|u\|_{L^s(\mathbb{R}^n)} \le C\|(\Delta_{\mathbb{R}^n} + \zeta)u\|_{L^r(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

where $\Delta_{\mathbb{R}^n} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ denotes the standard Laplacian on Euclidean space. The first condition in (1.1) is dictated by scaling and the other part of (1.1) was also shown in [7] to be necessary for (1.2).

Euclidean space of course is the unique simply connected manifold of constant curvature equal to zero. The purpose of this paper is to prove sharp theorems for simply connected manifolds of constant curvature either $+\kappa$ or $-\kappa$, $\kappa > 0$, which naturally imply (1.2) when $\kappa \searrow 0$. In the case of positive curvature $+\kappa$, we recall that the manifold is the sphere endowed with the metric which in geodesic polar coordinates $r\theta$, $\theta \in S^{n-1}$ about any point is given by

$$ds^2 = dr^2 + \left(\frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^2 |d\theta|^2, \quad 0 < r < \frac{\pi}{\sqrt{\kappa}}, \tag{1.3}$$

while in the case of constant negative curvature $-\kappa$, $\kappa > 0$, it is $\mathbb{R}^n$ endowed with the metric which in any geodesic normal coordinate system takes the form

$$ds^2 = dr^2 + \left(\frac{\sinh\sqrt{-\kappa}r}{\sqrt{-\kappa}}\right)^2 |d\theta|^2, \quad 0 < r < \infty, \tag{1.4}$$

(see e.g., [3]). The volume elements associated with these metrics then of course are

$$dV_\kappa = \left(\frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^{n-1} dr d\theta, \quad 0 < r < \frac{\pi}{\sqrt{\kappa}}, \ \kappa > 0, \tag{1.5}$$

and

$$dV_{-\kappa} = \left(\frac{\sinh\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^{n-1} dr d\theta, \quad 0 < r < \infty, \ -\kappa < 0, \tag{1.6}$$

respectively. The Laplacian associated with the metrics in these coordinates then is simply given by

$$\Delta_\kappa = \partial_r^2 + (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r)\partial_r + \left(\sqrt{\kappa}\csc(\sqrt{\kappa}r)\right)^2 \Delta_{S^{n-1}}, \quad \kappa > 0, \tag{1.7}$$

and

$$\Delta_{-\kappa} = \partial_r^2 + (n-1)\sqrt{\kappa}\coth(\sqrt{\kappa}r)\partial_r + \left(\sqrt{\kappa}\operatorname{csch}(\sqrt{\kappa}r)\right)^2 \Delta_{S^{n-1}}, \quad -\kappa < 0. \tag{1.8}$$

When $\kappa = 1$, we in the case of the standard round unit sphere and write $dV_{S^n} = dV_1$ and $\Delta_{S^n} = \Delta_1$, while for $\kappa = -1$, we are in standard hyperbolic space and write $dV_{-1} = dV_{\mathbb{H}^n}$ and $\Delta_{-1} = \Delta_{\mathbb{H}^n}$.