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Foundational aspects of singular integrals

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ABSTRACT

We investigate integration of classes of real-valued continuous functions on $(0,1]$. Of course difficulties arise if there is a non-L^1 element in the class, and the Hadamard finite part integral (*p.f.*) does not apply. Such singular integrals arise naturally in many contexts including PDEs and singular ODEs. The Lebesgue integral as well as *p.f.*, starting at zero, obey two fundamental conditions: (i) they act as antiderivatives and, (ii) if $f = g$ on $(0,a)$, then their integrals from 0 to x coincide for any $x \in (0,a)$.

We find that integrals from zero with the essential properties of *p.f.*, plus positivity, exist by virtue of the Axiom of Choice (AC) on all functions on $(0,1]$ which are $L^1((\varepsilon,1])$ for all $\varepsilon > 0$. However, this existence proof does not provide a satisfactory construction. Without some regularity at 0, the existence of general antiderivatives which satisfy only (i) and (ii) above on classes with a non-L^1 element is independent of ZF (the usual ZFC axioms for mathematics without AC), and even of ZFDC (ZF with the Axiom of Dependent Choice). Moreover we show that there is no mathematical description that can be proved (within ZFC or even extensions of ZFC with large cardinal hypotheses) to uniquely define such an antiderivative operator.

Such results are precisely formulated for a variety of sets of functions, and proved using methods from mathematical logic, descriptive set theory and analysis. We also analyze *p.f.* on

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analytic functions in the punctured unit disk, and make the connection to singular initial value problems.

1. Introduction

In this paper we investigate integration of classes of real-valued continuous functions on $(0, 1]$ and of analytic functions in the punctured unit disk.

Integrals of functions which are singular in the interior of the interval of integration are relatively well understood. A notable example is the Hilbert transform $(Hf)(x) = \pi^{-1}\int_{-\infty}^{\infty} f(s)(s-x)^{-1}ds$ where the integrand is L^1 for large s. Evidently, the integrand is in $L^1(\mathbb{R})$ iff f vanishes identically and in general Hf needs an interpretation. Defined as a Cauchy principal value integral, the domain of H is a set of Hölder continuous functions [29]. However, by a substantial argument, regularity is *not needed* to ensure that a natural *extension* of Hf exist: H is a bounded operator on L^p for any $p \in (1, \infty)$. By a theorem of Titchmarsh, for f in L^p as above, Hf exists pointwise everywhere [39]. H also extends as a bounded operator from L^1 into weak-L^1 [38]. The extension preserves the essential properties of H.

In contrast, only sufficient conditions are known for the existence one-sided singular integrals such as

$$\Gamma(\alpha)J^{\alpha} := \int_0^x s^{\alpha-1} f(s)ds \tag{1}$$

with f bounded and $\operatorname{Re}\alpha < 0$. These are frequently encountered in PDEs, in the analysis of differential and pseudodifferential operators, orthogonal polynomials and many other contexts. Although the history of (1) goes back to Liouville [26] and Riemann [31], its first systematic treatment is due to Hadamard who interpreted integrals of the type $\int_a^b f(s)(b-s)^{\alpha-1}ds$ arising in hyperbolic PDEs. In (1), sufficient smoothness of f,

$$f \in C^n\big((0, b]\big) \quad \text{and} \quad f^{(n)}(s)s^{\alpha+n-1} \in L^1 \tag{2}$$

allows for the Hadamard "partie finie" (*p.f.*, finite part, see Appendix A) to provide an extension of the usual integral. This extension, as shown by Hadamard has the properties (i) and (ii) above. In fact, it has all the properties of Lebesgue integration but positivity [16]. Riesz [32] showed that *p.f.* can be (essentially) equivalently defined by analytic continuation starting with $\operatorname{Re}\alpha > 0$; Schwartz (see [33] and [18]) reinterprets *p.f.* as a distributional regularization. See Appendix A for a brief review and further references. These reinterpretations make sense if (2) holds. In this paper, we use *Riesz's definition of p.f.*