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Partially harmonic forms and models of H-series

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A B S T R A C T

Let G be a real reductive Lie group, let $H = TA$ be the identity component of a Cartan subgroup, and let $\mathfrak{h}$ be the corresponding Cartan subalgebra. This leads to a parabolic subgroup of G whose identity component is MAN. The unitary G-representations induced by MAN are known as the H-series. We study symplectic geometry of $G \times \mathfrak{h}$ and apply geometric quantization to construct unitary G-representations by partially harmonic forms. They are direct integrals of the H-series, indexed by the image of the moment map. We also perform symplectic reduction and symplectic induction, and consider their analogues in representation theory via geometric quantization.

1. Introduction

Let G be a connected real reductive Lie group, let H be the identity component of a Cartan subgroup, and let $\mathfrak{h}$ be its Lie algebra. Let

$$X = G \times \mathfrak{h}.$$

We shall perform geometric quantization [14] to symplectic forms ω on X and construct unitary G-representations $\mathcal{H}(X, \omega)$. We show that $\mathcal{H}(X, \omega)$ is a direct integral of the H-series defined by Wolf [20,21], indexed by the image of the moment map of ω. According to Gelfand and Zelevinski [7], if G is compact, a *model* is a unitary G-representation

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on a Hilbert space in which every irreducible occurs once. In [2], we use $\mathcal{H}(X,\omega)$ to construct a model of G. For noncompact G, the definition of model can be generalized so that it consists of certain series of representations. In [3] (resp. [5]), we construct a model of discrete series (resp. principal series) when $\mathfrak{h}$ is a compact (resp. split) Cartan subalgebra. This article constructs a model of H-series for general $\mathfrak{h}$. Geometric quantization enables us to compare certain processes in symplectic geometry with their analogues in representation theory. We show that symplectic reduction [15] on (X,ω) corresponds to a direct integrand of $\mathcal{H}(X,\omega)$. We also introduce symplectic induction, which corresponds to induced representation. We now describe these projects in more details.

For convenience, we always assume that G is of Harish-Chandra class, namely its semisimple part (G,G) has finite center [13, Ch. VII-2]. Write $H = TA$, where T is a compact torus and A is diffeomorphic to the Euclidean space. The lower case Gothic letters always denote the Lie algebras, and subscript $\mathbb{C}$ always denotes complexification. For example $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$. Let MA be the identity component of the centralizer of A in G. By a choice of positive $\mathfrak{a}$-roots, we obtain the nilpotent subgroup N corresponding to the positive root spaces. Then MAN is the identity component of a parabolic subgroup of G. Since M has compact Cartan subgroup T, it has nonempty discrete series $\widehat{M}_{d.s.}$ [11]. Let $\Theta \in \widehat{M}_{d.s.}$ and $\nu \in \widehat{A}$. The H-series representation is

$$\mathrm{Ind}_{MAN}^{G}(\Theta \otimes \nu \otimes 1).$$

By the way, in [20,21], H is a Cartan subgroup of G, and MA is the centralizer of A in G; whereas our H and M are their identity components. So the resulting H-series differ slightly.

The left and right G-actions on G extend naturally to left and right G-actions on X by fixing $\mathfrak{h}$. There is also a right $\mathfrak{h}$-action on X, given by translation on the $\mathfrak{h}$-component of X. We call them the left action L of G and right action R of $G \times \mathfrak{h}$ on X, written as $G \times (G \times \mathfrak{h})$-actions.

In Section 2, we study the partially complex geometry of $G/N \times \mathfrak{h}$ and G/HN. These spaces are fibrations over $B = G/MAN$, and a choice of positive root system $\Delta^+(\mathfrak{m},\mathfrak{t})$ endows their fibers with canonical complex structures. We use these complex fibers to construct the partial Dolbeault differential forms Ω_B^q in (2.11). The Dolbeault operator $\bar{\partial}$ acts on $\Omega_B^\bullet$ to form a cochain complex. If in addition $\Omega_B^\bullet$ has an L^2-structure, let $\bar{\partial}^*$ be the formal adjoint on the square-integrable elements. By taking $\ker \bar{\partial} \cap \ker \bar{\partial}^*$ followed by Hilbert space completion, we obtain the partially harmonic forms $\mathcal{H}_B^q$.

Let $\Delta(\mathfrak{g},\mathfrak{h})$ and $\Delta(\mathfrak{m},\mathfrak{t})$ denote the roots of $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})$ and $(\mathfrak{m}_{\mathbb{C}},\mathfrak{t}_{\mathbb{C}})$ respectively. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Define the regular elements

$$\begin{aligned} \mathfrak{h}_{\mathrm{reg}} &= \bigl\{v \in \mathfrak{h};\ \alpha(v) \neq 0 \text{ for all } \alpha \in \Delta(\mathfrak{g},\mathfrak{h})\bigr\} + \mathfrak{z}, \\ \mathfrak{t}_{\mathrm{reg}} &= \bigl\{v \in \mathfrak{t};\ \alpha(v) \neq 0 \text{ for all } \alpha \in \Delta(\mathfrak{m},\mathfrak{t})\bigr\} + (\mathfrak{z} \cap \mathfrak{t}). \end{aligned} \tag{1.1}$$

If $u + v \in \mathfrak{h}_{\mathrm{reg}}$ where $u \in \mathfrak{t}$ and $v \in \mathfrak{a}$, then $u \in \mathfrak{t}_{\mathrm{reg}}$.