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Asymptotic expansions for trace functionals

Anna Skripka [1]

Department of Mathematics and Statistics, University of New Mexico, 400 Yale Blvd NE, MSC01 1115, Albuquerque, NM 87131, USA

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ABSTRACT

We obtain Taylor approximations for functionals $V \mapsto \mathrm{Tr}(f(H_0+V))$ defined on the bounded self-adjoint operators, where H_0 is a self-adjoint operator with compact resolvent and f is a sufficiently nice scalar function, relaxing assumptions on the operators made in [17], and derive estimates and representations for the remainders of these approximations.

1. Introduction

Let H_0 be an unbounded self-adjoint operator, V a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$, f a sufficiently nice scalar function, and let $f(H_0+V)$ be defined by the standard functional calculus. The functionals $f \mapsto \mathrm{Tr}(f(H_0+V))$ and $V \mapsto \mathrm{Tr}(f(H_0+V))$ or their modifications have been involved in problems of perturbation theory (of, for instance, differential operators) and noncommutative geometry since as early as 1950's (see, e.g., [1,2,4,6,7,9,12,17]). The latter functional in the context of noncommutative geometry is called the spectral (action) functional [6].

Assume that the resolvent of H_0 belongs to some Schatten ideal (or, more generally, $\mathrm{Tr}(e^{-tH_0^2}) < \infty$, for any $t > 0$), $\|\delta(V)\| < \infty$, $\|\delta^2(V)\| < \infty$, where $\delta(\cdot) = [|H_0|, \cdot]$, and f is a sufficiently nice even function. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of eigenvalues

E-mail address: skripka@math.unm.edu.

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of H_0 counting multiplicity and let $\{\psi_k\}_{k=1}^\infty$ be an orthonormal basis of the respective eigenvectors. The asymptotic expansion of the spectral action functional

$$\operatorname{Tr}\big(f(H_0+V)\big) = \operatorname{Tr}\big(f(H_0)\big) + \sum_{p=1}^\infty \frac{1}{p} \sum_{i_1,\dots,i_p} \big(f'\big)^{[p-1]}(\mu_{i_1},\dots,\mu_{i_p})\langle V\psi_{i_1},\psi_{i_2}\rangle \cdots \times \langle V\psi_{i_{p-1}},\psi_{i_p}\rangle\langle V\psi_{i_p},\psi_{i_1}\rangle, \tag{1.1}$$

where $(f')^{[p-1]}$ is the divided difference of order $p-1$ of the function f', was derived in [17], extending the results of [9] for finite-dimensional operators. (The precise assumptions on H_0, V, and f can be found in [17, Theorem 18].)

In this paper, we obtain the asymptotic expansion (1.1) under relaxed assumptions on H_0 and V and find bounds for the remainders of the respective approximations by taking a different approach to the problem. Specifically, we assume that $H_0 = H_0^*$ has compact resolvent, $V = V^* \in \mathcal{B}(\mathcal{H})$ (where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$), and f is a sufficiently nice compactly supported function (but no summability restriction on H_0 is made, H_0 is not assumed to be positive, and f is not assumed to be even). Let

$$R_{H_0,f,n}(V) := \operatorname{Tr}\big(f(H_0+V)\big) - \operatorname{Tr}\big(f(H_0)\big) - \sum_{p=1}^{n-1} \frac{1}{p} \sum_{i_1,\dots,i_p} \big(f'\big)^{[p-1]}(\mu_{i_1},\dots,\mu_{i_p})\langle V\psi_{i_1},\psi_{i_2}\rangle \cdots \times \langle V\psi_{i_{p-1}},\psi_{i_p}\rangle\langle V\psi_{i_p},\psi_{i_1}\rangle. \tag{1.2}$$

In Theorem 3.4 and Corollary 3.5, we establish the bound

$$\big|R_{H_0,f,n}(V)\big| = \mathcal{O}\big(\|V\|^n\big) \tag{1.3}$$

and find an explicit estimate for $\mathcal{O}(\|V\|^n)$ in Theorem 3.2 and Remark 3.3(i). (The case $n=1$ also follows from [1].) If, in addition, H_0 has Hilbert–Schmidt resolvent, we refine the bound (1.3) of Theorem 3.4 in Theorem 3.8. In Theorem 3.10, we show that the functional $C_c^3(\mathbb{R}) \ni f'' \mapsto R_{H_0,f,2}(V)$ is given by a locally finite absolutely continuous measure. (An analogous result for the functional $f' \mapsto R_{H_0,f,1}(V)$ was obtained in [1].)

2. Preliminaries

The asymptotic expansion (1.1) can be rewritten as

$$\operatorname{Tr}\big(f(H_0+V)\big) = \operatorname{Tr}\big(f(H_0)\big) + \sum_{p=1}^\infty \frac{1}{p} \sum_{\lambda_1,\dots,\lambda_p\in\operatorname{spec}(H)} \big(f'\big)^{[p-1]}(\lambda_1,\dots,\lambda_p) \times \operatorname{Tr}\big(E_{H_0}(\lambda_1)V\cdots VE_{H_0}(\lambda_p)V\big), \tag{2.1}$$

where E_{H_0} is the spectral measure of $H_0 = H_0^*$.