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Chaotic dynamics of the heat semigroup on Riemannian symmetric spaces

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ABSTRACT

We show that the heat semigroup generated by certain perturbations of the Laplace–Beltrami operator on the Riemannian symmetric spaces of noncompact type is *chaotic* on their L^p-spaces when $2 < p < \infty$. Both the range of p and the range of chaos-inducing perturbation are sharp. This extends a result of Ji and Weber [17] where it was shown that under identical conditions the heat operator is *subspace-chaotic* on these spaces.

1. Introduction

The study of chaotic dynamical systems arising in geometric contexts is an emerging field that explores problems rooted in several classical areas of mathematics, such as ergodic theory, geometry and analysis. The object of study in this article is the natural occurrence of such a system in a general Riemannian symmetric space X of noncompact type and arbitrary rank, equipped with a Laplace–Beltrami operator $-\Delta$ endowed by its Riemannian structure. Such spaces are widely studied for their rich blend of geometric and analytic properties, combined with certain characteristic phenomena that are strikingly different from their Euclidean or compact counterparts. Information on such spaces

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necessary for this article has been gathered in Section 3, but more exhaustive treatments may be found in [14,10]. A fundamental example of this class is the hyperbolic n-space, which may orient the reader less familiar with this general setting.

We obtain a complete characterization of the chaotic behaviour of shifts of the heat semigroup, namely

$$T_t = e^{-t(\Delta - c)}, \quad t \geqslant 0,\ c \in \mathbb{R}, \tag{1.1}$$

on the Lebesgue spaces $L^p(X)$. The ergodic–theoretic terminology used in this paper (such as *chaotic*, *subspace-chaotic*, *hypercyclic* and *periodic*) as well as many properties of T_t are standard in the literature, but for completeness the precise definitions and relevant results have been included in Sections 2 and 3. Our results are optimal in the exponent p and in the shift parameter c, and extends recent work of Ji and Weber [17], where chaotic behaviour was shown to exist on a subspace of $L^p(X)$. The result of Ji and Weber stated in our notation is the following. As mentioned before, all terminology used in the statement have been explained in Sections 2 and 3.

Theorem 1.1. *(See [17, Theorems 3.1, 3.2, Corollary 3.3].) Let X be a Riemannian symmetric space of noncompact type. Let $\mathbb{B}_p$ be the subspace of K bi-invariant functions in $L^p(X)$ and $\Delta^{\#}$ be the restriction of Δ to $\mathbb{B}_p$.*

(a) *Suppose that X is of rank one. If $p > 2$ then there exists a $c_p > 0$ such that for all $c > c_p$, $T_t : \mathbb{B}_p \to \mathbb{B}_p$ is chaotic. Hence T_t is subspace chaotic on $L^p(X)$. If $p = 2$ (respectively $1 < p < 2$) then it is not chaotic (respectively not hypercyclic) for any $c \in \mathbb{R}$.*
(b) *Suppose that the rank of X is higher than one. If $p > 2$ then there exists a $c_p > 0$ such that for all $c > c_p$, $T_t : \mathbb{B}_p \to \mathbb{B}_p$ is subspace chaotic. In the higher rank case it is only known that there is an invariant subspace of $\mathbb{B}_p$ on which T_t is chaotic.*

To amplify further, in [17] the authors work with a specific subspace $\mathbb{B}_p$ of $L^p(X)$, namely the space of K bi-invariant functions in $L^p(X)$. Chaos is shown to exist when $p > 2$

- on $\mathbb{B}_p$ if X is of rank one, and
- on an unspecified subspace of $\mathbb{B}_p$ for X of arbitrary rank.

This is done partly by an application of Theorem 3.1 in [7], which relies on the construction of an auxiliary $\mathbb{B}_p$-valued function F on some open subset $\Omega \subseteq \sigma_{\mathrm{pt}}(\Delta) \subseteq \mathbb{C}$ such that $z \mapsto \phi(F(z))$ is a nontrivial analytic function on Ω for every nonzero $\phi \in \mathbb{B}^*$. In [17], an intermediate step in this construction is to create a function from the set Ω to $\mathbb{C}^n$ whose image has nonempty interior. This method works when $n = 1$ (the rank one case), but fails for general n due to an obvious dimensional disparity. To get around this for $n > 1$, [17] employs a result due to Banasiak and Moszyński [4] generalizing the