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Uniformly factoring weakly compact operators

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ABSTRACT

Let X and Y be separable Banach spaces. Suppose Y either has a shrinking basis or Y is isomorphic to $C(2^{\mathbb{N}})$ and $\mathcal{A}$ is a subset of weakly compact operators from X to Y which is analytic in the strong operator topology. We prove that there is a reflexive space with a basis Z such that every $T \in \mathcal{A}$ factors through Z. Likewise, we prove that if $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$ is a set of operators whose adjoints have separable range and is analytic in the strong operator topology then there is a Banach space Z with separable dual such that every $T \in \mathcal{A}$ factors through Z. Finally we prove a uniform version of this result in which we allow the domain and range spaces to vary.

1. Introduction

Recall that if X and Y are Banach spaces then a bounded operator $T : X \to Y$ is called *weakly compact* if $\overline{T(B_X)}$ is weakly compact, where B_X is the unit ball of X. If there exists a reflexive Banach space Z and bounded operators $T_1 : X \to Z$ and $T_2 : Z \to Y$ with $T = T_2 \circ T_1$ then T_1 and T_2 are both weakly compact by Alaoglu's

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theorem and hence $T : X \to Y$ is weakly compact as well. Thus it is immediate that any bounded operator which factors through a reflexive Banach space is weakly compact. In their seminal 1974 paper [11], Davis, Figiel, Johnson and Pełczyński proved that the converse is true as well. That is, every weakly compact operator factors through a reflexive Banach space. Likewise, every bounded operator whose adjoint has separable range factors through a Banach space with separable dual. Using the DFJP interpolation technique, in 1988 Zippin proved that every separable reflexive Banach space embeds into a reflexive Banach space with a basis and that every Banach space with separable dual embeds into a Banach space with a shrinking basis [30].

For each separable reflexive Banach space X we may choose a reflexive Banach Z with a basis such that X embeds into Z. It is natural to consider when the choice of Z can be done uniformly. That is, given a set of separable reflexive Banach spaces $\mathcal{A}$, when does there exist a reflexive Banach space Z with a basis such that X embeds into Z for every $X \in \mathcal{A}$? Szlenk proved that there does not exist a Banach space Z with separable dual such that every separable reflexive Banach space embeds into Z [29]. Bourgain proved further that if Z is a separable Banach space such that every separable reflexive Banach space embeds into Z then every separable Banach space embeds into Z [9]. Thus, any uniform embedding theorem must consider strict subsets of the set of separable reflexive Banach spaces. In his Phd thesis, Bossard developed a framework for studying sets of Banach spaces using descriptive set theory [8,7]. In this context, it was shown in [14] and [28] that if $\mathcal{A}$ is an analytic set of separable reflexive Banach spaces then there exists a separable reflexive Banach space Z such that X embeds into Z for all $X \in \mathcal{A}$, and in [14] and [16] it was shown that if A is an analytic set of Banach spaces with separable dual then there exists a Banach space Z with separable dual such that X embeds into Z for all $X \in \mathcal{A}$. In particular, solving an open problem posed by Bourgain [9], there exists a separable reflexive Banach space Z such that every separable uniformly convex Banach space embeds into Z [27]. As the set of all Banach spaces which embed into a fixed Banach space is analytic in the Bossard framework, these uniform embedding theorems are optimal.

The goal for this paper is to return to the original operator factorization problem with the same uniform perspective that was applied to the embedding problems. That is, given separable Banach spaces X and Y and a set of weakly compact operators $\mathcal{A} \subset \mathcal{L}(X, Y)$, we want to know when does there exist a reflexive Banach space Z such that T factors through Z for all $T \in \mathcal{A}$. We are able to answer this question in the following cases.

Theorem 1. *Let X and Y be separable Banach spaces and let $\mathcal{A}$ be a set of weakly compact operators from X to Y which is analytic in the strong operator topology. Suppose either Y has a shrinking basis or Y is isomorphic to $C(2^{\mathbb{N}})$. Then there is a reflexive Banach space Z with a basis such that every $T \in \mathcal{A}$ factors through Z.*

Theorem 2. *Let X be a separable Banach space and let $\mathcal{A} \subset \mathcal{L}(X, C(2^{\mathbb{N}}))$ be a set of bounded operators whose adjoints have separable range which is analytic in the strong*