Journal of Functional Analysis 266 (2014) 2944–2958

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Sharp Sobolev inequalities in Lorentz spaces for a mean oscillation

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A R T I C L E   I N F O

Article history:
Received 1 May 2013
Accepted 25 December 2013
Available online 17 January 2014
Communicated by H. Brezis

Keywords:
Sobolev inequalities in Lorentz spaces
Optimal constant
Critical Hardy inequalities
Scale invariance property

A B S T R A C T

We exhibit the optimal constant for Sobolev inequalities in Lorentz spaces for a mean oscillation, and its relation with a boundedness of the Hardy–Littlewood maximal operator in Sobolev spaces. Some applications to a scale invariant form of Hardy's inequality in a limiting case are also considered.

1. Introduction and main results

1.1. Sobolev inequalities in Lorentz spaces for a mean oscillation

The standard Sobolev inequality states that, if $n \geqslant 2$ and $1 \leqslant p < n$, then

$$S_{n,p}\|u\|_{L^{p^*}} \leqslant \|\nabla u\|_{L^p} \tag{1.1}$$

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0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2013.12.023

for every $u \in W^{1,p}(\mathbb{R}^n)$, where p^* is the Sobolev conjugate number defined by $p^* = np/(n-p)$ and $S_{n,p}$ is the optimal constant for the inequality (1.1) given by

$$S_{n,p} = \sqrt{\pi} n^{\frac{1}{p}} \left(\frac{n-p}{p-1}\right)^{\frac{p-1}{p}} \left[\frac{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p})}{\Gamma(n)\Gamma(1+\frac{n}{2})}\right]^{\frac{1}{n}}.$$

The inequality (1.1) with the optimal constant was proved by Federer and Fleming [20] and Maz'ya [30] for $p=1$ and by Aubin [4] and Talenti [36] for $1<p<n$. The inequality (1.1) is also called a Sobolev embedding theorem since (1.1) implies the embedding

$$W^{1,p}\big(\mathbb{R}^n\big) \subset L^{p^*}\big(\mathbb{R}^n\big). \tag{1.2}$$

This embedding is known as the optimal embedding in the framework of Lebesgue spaces, i.e. the smallest Lebesgue space which contains $W^{1,p}(\mathbb{R}^n)$ is $L^{p^*}(\mathbb{R}^n)$. More precisely, (1.2) is the optimal embedding in the framework of Orlicz spaces, which is a natural generalization of Lebesgue spaces (see [11]). However, Peetre [34] proved that the Sobolev embedding (1.2) can be improved within the framework of Lorentz spaces (see also [38]).

Lorentz spaces are known as real interpolation spaces between Lebesgue spaces and can be defined via the notion of Schwarz symmetrization. Let ϕ be a measurable function on $\mathbb{R}^n$ whose level sets have finite measure for every level. Then the function

$$\mu(\lambda) := \big|\big\{x\colon\ \big|\phi(x)\big| > \lambda\big\}\big|, \quad \lambda \geqslant 0,$$

is the distribution function of ϕ and

$$\phi^\sharp(r) := \inf\big\{\lambda > 0\colon\ \mu(\lambda) \leqslant |B_r|\big\}$$

is radially symmetric and non-increasing rearrangement of ϕ, where B_r is the ball centered at the origin with radius r and $|A|$ is the n-dimensional Lebesgue measure of $A \subset \mathbb{R}^n$. We call the function $\phi^\sharp(|x|)$, $x \in \mathbb{R}^n$ as Schwarz symmetrization of ϕ. Then, we define Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ as

$$L^{p,q}\big(\mathbb{R}^n\big) := \Bigg\{u\colon\ \text{measurable in } \mathbb{R}^n \ \Bigg| \ \|u\|_{L^{p,q}} := |B_1|^{\frac{q-p}{pq}} \Bigg(\int_{\mathbb{R}^n} \big(|x|^{\frac{n}{p}} u^\sharp\big(|x|\big)\big)^q \frac{dx}{|x|^n}\Bigg)^{\frac{1}{q}} < \infty\Bigg\}.$$

A sharpened version of the Sobolev inequality with the optimal constant states that, if $n \geqslant 2$ and $1 \leqslant p < n$, then

$$\left(\frac{\sqrt{\pi}}{\Gamma(1+\frac{n}{2})^{\frac{1}{n}}}\right)\frac{n-p}{p}\|u\|_{L^{p^*,p}} \leqslant \|\nabla u\|_{L^p} \tag{1.3}$$