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On convergence rates in approximation theory for operator semigroups ☆

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Abstract

We create a new, functional calculus, approach to approximation formulas for C_0-semigroups on Banach spaces restricted to the domains of fractional powers of their generators. This approach allows us to equip the approximation formulas with rates which appear to be optimal in a natural sense. In the case of analytic semigroups, we improve our general results obtaining better convergence rates which are optimal in that case too. The setting of analytic semigroups includes also the case of convergence on the whole space. As an illustration of our approach, we deduce optimal convergence rates in classical approximation formulas for C_0-semigroups restricted to the domains of fractional powers of their generators.

Keywords: Bernstein functions; Approximation of C_0-semigroups; Functional calculus; Convergence rates; Banach spaces; Interpolation

1. Introduction

Approximation theory is a classical chapter in the theory of C_0-semigroups with various applications to PDEs and their numerical analysis. By approximating a C_0-semigroup with exponentials of bounded operators or with rational functions of its generator one can often reduce

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the study of a difficult problem to a simpler one. An instance of such an approach is the famous Hille–Yosida generation theorem where the Yosida approximation arises.

The two core results in approximation theory, the Trotter–Kato theorem and the Chernoff product formula (see e.g. [16, Chapter III.4] and [16, Chapter III.5, a)] respectively), proved to be very helpful in many areas of analysis, including differential operators, mathematical physics and probability theory. The following particular cases of these results representing different approaches to semigroup approximation became well-known and found their way into most of books on semigroup theory, see e.g. [16, Chapters III.4, III.5], [20, Chapters 1.7, 1.8], [35, Chapters 3.4–3.6], [10, Chapter 5]. Note that a) and b) below follow from the Trotter–Kato approximation theorem, while c) can be derived from Chernoff's product formula, see e.g. [16, pp. 214–216 and p. 223].

Theorem 1.1. *Let $(e^{-tA})_{t\geqslant 0}$ be a bounded C_0-semigroup on a Banach space X. Then the following statements hold.*

a) [Yosida approximation] *For every $x \in X$,*

$$e^{-tA}x = \lim_{n\to\infty} e^{-ntA(n+A)^{-1}}x$$

uniformly in t from compacts in $\mathbb{R}_+$.

b) [Dunford–Segal approximation] *For every $x \in X$,*

$$e^{-tA}x = \lim_{n\to\infty} e^{-nt(1-e^{-A/n})}x$$

uniformly in t from compacts in $\mathbb{R}_+$.

c) [Euler approximation] *For every $x \in X$,*

$$e^{-tA}x = \lim_{n\to\infty} (1+tA/n)^{-n}x$$

uniformly in t from compacts in $\mathbb{R}_+$.

(The name for the approximation formula in b) is not well-established, although some authors use this terminology. We find it natural too since the formula was introduced for the first time by Dunford and Segal in [13].)

While theorems on semigroup approximation are very useful, with a few exceptions, they still have a merely qualitative character and the natural problem of finding optimal rates of approximation remains open. The aim of our paper is to fill this gap.

The approximations introduced in Theorem 1.1 will be of primary importance for us. So, before describing our approach, we give a short account of known rate estimates for these approximations. Among the three, the Euler approximation attracted most attention and relevant results can be summarized as follows.

Theorem 1.2. *Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\geqslant 0}$ on a Banach space X. Then the following hold.*