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Bulk asymptotics for polyanalytic correlation kernels

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Abstract

For a weight function $Q : \mathbb{C} \to \mathbb{R}$ and a positive scaling parameter m, we study reproducing kernels $K_{q,mQ,n}$ of the polynomial spaces

$$A^2_{q,mQ,n} := \operatorname{span}_{\mathbb{C}}\{\bar{z}^r z^j \mid 0 \leqslant r \leqslant q-1,\ 0 \leqslant j \leqslant n-1\}$$

equipped with the inner product from the space $L^2(e^{-mQ(z)}\,\mathrm{d}A(z))$. Here $\mathrm{d}A$ denotes a suitably normalized area measure on $\mathbb{C}$. For a point z_0 belonging to the interior of certain compact set $\mathcal{S}$ and satisfying $\Delta Q(z_0) > 0$, we define the rescaled coordinates

$$z = z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, \qquad w = z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}.$$

The following universality result is proved in the case $q = 2$:

$$\frac{1}{m\Delta Q(z_0)}\big|K_{q,mQ,n}(z,w)\big|e^{-\frac{1}{2}mQ(z)-\frac{1}{2}mQ(w)} \to \big|L^1_{q-1}\big(|\xi-\lambda|^2\big)\big|e^{-\frac{1}{2}|\xi-\lambda|^2}$$

as $m, n \to \infty$ while $n \geqslant m - M$ for any fixed $M > 0$, uniformly for (ξ, λ) in compact subsets of $\mathbb{C}^2$. The notation L^1_{q-1} stands for the associated Laguerre polynomial with parameter 1 and degree $q-1$. This generalizes a result of Ameur, Hedenmalm and Makarov concerning analytic polynomials to *bianalytic polynomials*. We also discuss how to generalize the result to $q > 2$. Our methods include a simplification of a Bergman kernel expansion algorithm of Berman, Berndtsson and Sjöstrand in the one compex variable

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setting, and extension to the context of polyanalytic functions. We also study off-diagonal behaviour of the kernels $K_{q,mQ,n}$.

Keywords: Polyanalytic function; Determinantal point process; Landau level; Bergman kernel

1. Introduction

1.1. Notation

We will write ∂X and $\mathrm{int}(X)$ for the boundary and the interior of a subset X of the complex plane $\mathbb{C}$. By 1_X we mean the characteristic function of the set X. We let

$$\mathrm{d}A(z) = \pi^{-1}\,\mathrm{d}x\,\mathrm{d}y, \quad \text{where } z = x + \mathrm{i}y \in \mathbb{C},$$

be the normalized area measure in $\mathbb{C}$, and use the standard Wirtinger derivatives

$$\partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y), \qquad \bar{\partial}_z := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y).$$

We will often omit the subscripts if there is no risk of confusion. We write $\Delta = \partial\bar{\partial}$, and it can be observed that this equals to one quarter of the usual Laplacian. We write $\mathbb{D}$ for the open unit disk, and more generally $\mathbb{D}(z, r)$ for the disk with center z and radius r. Given a Lebesgue measurable function $w : \mathbb{C} \to \mathbb{R}$, we denote by $L^2(w)$ the space of measurable functions $\mathbb{C} \to \mathbb{C}$ which are square-integrable with respect to the measure $w(z)\,\mathrm{d}A(z)$.

1.2. Spaces of polyanalytic polynomials

Let $Q : \mathbb{C} \to \mathbb{R}$ be a continuous function satisfying

$$Q(z) \geqslant (1+\epsilon)\log|z|^2, \quad |z| \geqslant C \tag{1.1}$$

for two positive numbers ϵ and C. This function will be referred to as the *weight*. We set

$$\mathrm{Pol}_{q,n} := \mathrm{span}_{\mathbb{C}}\big\{\bar{z}^r z^j \;\big|\; 0 \leqslant r \leqslant q-1,\ 0 \leqslant j \leqslant n-1\big\},$$

and

$$A^2_{q,mQ,n} := \mathrm{Pol}_{q,n} \cap L^2\big(e^{-mQ(z)}\big).$$

The space $A^2_{q,mQ,n}$ is a finite dimensional, and thus closed, subspace of $L^2(e^{-mQ(z)})$. We see that when $m \geqslant n+q-1$, the growth condition on Q implies that $A^2_{q,mQ,n}$ contains the whole $\mathrm{Pol}_{q,n}$.

Notice that $A^2_{1,mQ,n}$ consists of analytic polynomials of degree at most $n-1$. For a more general $q \geqslant 1$, functions in the spaces $A^2_{q,mQ,n}$ will be called *q-analytic polynomials*.