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Global in time estimates for the spatially homogeneous Landau equation with soft potentials

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Abstract

This paper deals with some global in time a priori estimates of the spatially homogeneous Landau equation for soft potentials $\gamma \in [-2, 0)$. For the first result, we obtain the estimate of weak solutions in $L_t^\alpha L_v^{3-\varepsilon}$ for $\alpha = \frac{2(3-\varepsilon)}{3(2-\varepsilon)}$ and $0 < \varepsilon < 1$, which is an improvement over estimates by Fournier and Guerin [10]. For the second result, we have the estimate of weak solutions in $L_t^\infty L_v^p$, $p > 1$, which extends part of results by Fournier and Guerin [10] and Alexandre, Liao and Lin [1]. As an application, we deduce some global well-posedness results for $\gamma \in [-2, 0)$. Our estimates include the case $\gamma = -2$, which is the key point in this paper.

Keywords: Entropy production; Landau equation; Soft potentials; Weak solutions

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1. Introduction

1.1. The Landau equations

We consider the spatially homogeneous Landau equation in dimension three for soft potentials. This equation of kinetic physics, also called Fokker–Planck–Landau equation, has been derived from the Boltzmann equation when the grazing collisions prevail in the gas. It describes the evolution of the density function $f_t(v)$ of particles having the velocity $v \in \mathbb{R}^3$ at time $t \geqslant 0$:

$$\frac{\partial f_t(v)}{\partial t} = Q(f_t, f_t)(v), \tag{1}$$

with the collision operator

$$Q(f_t, f_t)(v) = \nabla_v \cdot \left\{ \int a(v - v_*)\left[f_t(v_*)\nabla f_t(v) - f_t(v)\nabla_* f_t(v_*)\right] dv_* \right\},$$

where $a(v)$ is a symmetric non-negative matrix, depending on a parameter $\gamma \in [-3, 1]$,

$$a(v) = |v|^{\gamma+2}\mathbf{P}(v),$$

and $\mathbf{P}(v)$ is the 3 by 3 matrix

$$\mathbf{P}(v) = I_3 - \frac{v \otimes v}{|v|^2}.$$

This leads to the usual classification in terms of hard potentials $\gamma > 0$, Maxwellian molecules $\gamma = 0$, soft potentials $\gamma \in [-2, 0)$, very soft potentials $\gamma \in (-3, -2)$ and Coulomb potential $\gamma = -3$. Just as for the Boltzmann equation, little is known for soft potentials, i.e. $\gamma < 0$, and even less for very soft potentials, i.e. $\gamma < -2$. In particular, $\gamma = -3$ corresponds to the important Coulombic interaction in plasma physics. Unfortunately, it is also the most difficult case to study. However, the Landau equation can be derived from the Boltzmann equation with $\gamma \in (-3, -1)$. Note the fact that the more γ is negative, the more the Landau equation is physically interesting. In this paper, we focus on soft potentials $\gamma \in [-2, 0)$.

For a given non-negative initial data $f_{\mathrm{in}}(v)$, we shall use the notations

$$m(f_{\mathrm{in}}) = \int f_{\mathrm{in}}(v)\, dv, \qquad e(f_{\mathrm{in}}) = \frac{1}{2}\int f_{\mathrm{in}}(v)|v|^2\, dv, \qquad H(f_{\mathrm{in}}) = \int f_{\mathrm{in}}(v)\log f_{\mathrm{in}}(v)\, dv,$$

for the initial mass, energy and entropy. It is classical that if $f_{\mathrm{in}}(v) \geqslant 0$ and $m(f_{\mathrm{in}})$, $e(f_{\mathrm{in}})$, $H(f_{\mathrm{in}})$ are finite, then $f_{\mathrm{in}}(v)$ belongs to

$$L\log L\left(\mathbb{R}^3\right) = \left\{ f(v) \in L^1\left(\mathbb{R}^3\right):\ \int \left|f(v)\right|\left|\log\left(\left|f(v)\right|\right)\right| dv < \infty \right\}.$$

The solution of the Landau equation (1) satisfies, at least formally, the conservation of mass, momentum and kinetic energy, that is, for any $t \geqslant 0$,

$$\int f_t(v)\varphi(v)\, dv = \int f_{\mathrm{in}}(v)\varphi(v)\, dv, \quad \text{for } \varphi(v) = 1, v, |v|^2. \tag{2}$$