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The resolvent algebra: Ideals and dimension

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Abstract

Let (X, σ) be a symplectic space admitting a complex structure and let $\mathcal{R}(X, \sigma)$ be the corresponding resolvent algebra, *i.e.* the C*-algebra generated by the resolvents of selfadjoint operators satisfying canonical commutation relations associated with (X, σ). In previous work this algebra was shown to provide a convenient framework for the analysis of quantum systems. In the present article its mathematical properties are elaborated with emphasis on its ideal structure. It is shown that $\mathcal{R}(X, \sigma)$ is always nuclear and, if X is finite dimensional, also of type I (postliminal). In the latter case $\dim(X)$ labels the isomorphism classes of the corresponding resolvent algebras. For X of arbitrary dimension, principal ideals are identified which are the building blocks for all other ideals. The maximal and minimal ideals of the resolvent algebra are also determined.

Keywords: Resolvent algebra; Symplectic space; Canonical commutation relations; C*-algebra; Ideal; Representation; Isomorphism class

1. Introduction

In [5] we have defined and analyzed the resolvent algebra of the canonical commutation relations. Apart from the applications in that paper, this algebra has already demonstrated its usefulness elsewhere. For example, on its basis one can model in a C*-context superderivations which occur in supersymmetry, cf. [4], as well as in BRST-constraint theory, cf. [7]. It occurs also naturally in the representation theory of abelian Lie algebras of derivations acting on a C*-algebra [6]. Here we continue our analysis of the resolvent algebra, with particular emphasis on its ideal structure.

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We review the background which motivates the study of the resolvent algebra. Canonical systems of operators have always been a central ingredient in the modeling of quantum systems. These systems of operators may all be presented in the following general form: there is a real linear map ϕ from a given symplectic space (X, σ) to a linear space of selfadjoint operators on some common dense invariant core $\mathcal{D}$ in a Hilbert space $\mathcal{H}$, satisfying the relations

$$\left[\phi(f), \phi(g)\right] = i\sigma(f, g)\mathbf{1}, \qquad \phi(f)^* = \phi(f) \quad \text{on } \mathcal{D}.$$

In the case that X is finite dimensional, one can reinterpret this relation in terms of the familiar quantum mechanical position and momentum operators, and if X consists of Schwartz functions on some manifold one may consider ϕ to be a bosonic quantum field. The observables of the system are then constructed from the operators $\{\phi(f)\colon\ f \in X\}$, usually as polynomial expressions. Since one wants to study a variety of representations of such systems, it is convenient to cast the algebraic information of the canonical systems into C*-algebras, given the rich source of mathematical tools available there.

The obvious way to take this step is to form suitable bounded functions of the generically unbounded fields $\phi(f)$. In the approach introduced by Weyl, this is done by considering the C*-algebra generated by the set of unitaries

$$\left\{\exp\bigl(i\phi(f)\bigr)\colon\ f \in X\right\}.$$

Regarded as an abstract algebra, it is the familiar Weyl algebra [12], denoted by $\mathcal{W}(X, \sigma)$. The Weyl algebra suffers, however, from several well-known flaws with regard to physics. First and foremost, it does not admit the definition of much interesting dynamics (one-parameter automorphism groups), cf. [5,9]. Second, natural observables such as bounded functions of the Hamiltonian are not in $\mathcal{W}(X, \sigma)$. Third, the Weyl algebra has a vast number of representations in which representers of its generators $\phi(f)$ cannot be defined [1,10,11]. These nonregular representations describe situations where the field ϕ has "infinite strength". Whilst this is sometimes useful for idealizations, cf. for example the discussion of plane waves in [1] or of quantum constraints in [11], the majority of nonregular representations of the Weyl algebra is of no interest.

This motivates the consideration of alternative C*-algebraic versions of the canonical commutation relations. Instead of taking the C*-algebra generated by exponentials of the underlying generators, as for the Weyl algebra, we propose to consider the C*-algebra generated by their resolvents [5]. These are given on the underlying Hilbert space $\mathcal{H}$ by

$$\left\{R(\lambda, f) \doteq \bigl(i\lambda\mathbf{1} - \phi(f)\bigr)^{-1}\colon\ \lambda \in \mathbb{R}\backslash\{0\},\ f \in X\right\}.$$

All algebraic properties of the fields can be expressed in terms of relations amongst these resolvents and this fact allows one to define the unital C*-algebra generated by the resolvents also in representation independent terms. The structure of the resulting resolvent algebra will be studied below.

In contrast to the Weyl algebra, which is simple [3, Thm. 5.2.8], the resolvent algebra has ideals. This feature agrees with the observation that a unital C*-algebra which admits the definition of a sufficiently diverse variety of dynamics cannot be simple [5, p. 2767]. In the present article we analyze the ideal structure of the resolvent algebra and show that it depends sensitively on the size of the underlying quantum system. By a study of its primitive ideals we find that the