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Radial positive solutions of elliptic systems with Neumann boundary conditions

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Abstract

We consider radial solutions of elliptic systems of the form

$$\begin{cases} -\Delta u + u = a(|x|) f(u, v) & \text{in } B_R, \\ -\Delta v + v = b(|x|) g(u, v) & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial B_R, \end{cases}$$

where essentially a, b are assumed to be radially nondecreasing weights and f, g are nondecreasing in each component. With few assumptions on the nonlinearities, we prove the existence of at least one couple of nondecreasing nontrivial radial solutions. We emphasize that we do not assume any variational structure nor subcritical growth on the nonlinearities. Our result covers systems with supercritical as well as asymptotically linear nonlinearities.

Keywords: Elliptic system; Supercritical growth; Radial solutions; Neumann problem

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1. Introduction

Let B_R be the ball of radius R in $\mathbb{R}^N$, with $N \geqslant 2$. We consider the Neumann problem

$$\begin{cases} -\Delta u + u = a\big(|x|\big) f(u, v) & \text{in } B_R, \\ -\Delta v + v = b\big(|x|\big) g(u, v) & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial B_R, \end{cases} \tag{1}$$

where a, b and f, g satisfy the assumptions

(A) $a, b \in L^1(0, R)$ are nonnegative, nondecreasing and not identically zero;
(H1) $f, g \in C(\mathbb{R}^+ \times \mathbb{R}^+)$ are nonnegative and nondecreasing in each variable;
(H2)

$$\lim_{s+t \to 0^+} \frac{f(s,t) + g(s,t)}{s+t} = 0, \qquad \lim_{s+t \to +\infty} \frac{f(s,t) + g(s,t)}{s+t} = +\infty.$$

This class of systems includes for instance gradient type systems

$$\begin{cases} -\Delta u + u = \partial_u G\big(|x|, u, v\big) & \text{in } B_R, \\ -\Delta v + v = \partial_v G\big(|x|, u, v\big) & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial B_R, \end{cases} \tag{2}$$

with, for example, $G(|x|, u, v) = a(|x|) g(u, v)$, or Hamiltonian type systems

$$\begin{cases} -\Delta u + u = \partial_v H\big(|x|, u, v\big) & \text{in } B_R, \\ -\Delta v + v = \partial_u H\big(|x|, u, v\big) & \text{in } B_R, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial B_R, \end{cases} \tag{3}$$

where, e.g., $H(|x|, u, v) = a(|x|) F(v) + b(|x|) G(u)$, as well as nonvariational systems, namely systems which are not the Euler–Lagrange equations of an energy functional.

When dealing with (2), it is usually assumed that $G(s, t)$ grows at most like $|s|^p + |t|^q$ where $p, q < 2^* := 2N/(N-2)$. Such an assumption, which is referred to as a subcriticality condition, gives the required compactness to allow a study of the system through critical point theory. In the case of (3), the subcriticality assumption, which brings the required compactness, takes a relaxed form. Assuming as a paradigm that $H(s, t) = s^p + t^q$, the subcriticality condition [11,13] writes

$$\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}. \tag{4}$$

When dealing with nonvariational systems with topological methods, one also requires some compactness. This compactness usually corresponds to a priori bounds on a class of auxiliary systems associated to the original one. Growth limitations on f and g then appear as a main assumption to derive these a priori bounds. We refer to de Figueiredo [7] for further details.

Our main purpose in this work is to show that when looking for radial solutions of the Neumann problem (1), one can obtain existence results for broader classes of nonlinearities