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Uniform bounds for convolution and restricted X-ray transforms along degenerate curves

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ABSTRACT

We establish endpoint Lebesgue space bounds for convolution and restricted X-ray transforms along curves satisfying fairly minimal differentiability hypotheses, with affine and Euclidean arclengths. We also explore the behavior of certain natural interpolants and extrapolants of the affine and Euclidean versions of these operators.

1. Introduction

This article deals with the basic problem of determining the precise amount of L^p-improving for certain weighted averaging operators associated to curves in $\mathbb{R}^d$. In the unweighted case, this problem has been studied by Tao and Wright in wide generality, and in [22], they completely describe (except for boundary points) the set of (p, q) for which these operators map L^p boundedly into L^q, under certain smoothness hypotheses and in the presence of a cutoff. This set of (p, q) depends on the torsion (and appropriate generalizations thereof), but if instead the average is taken against an 'affine arclength measure,' the effects of vanishing torsion are mitigated, and the (p, q) region is larger.

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In fact (excepting boundary points), the new region is essentially independent of the curves [20].

We are interested in the questions of whether the endpoint estimates hold, whether there is a natural way to relate the weighted and unweighted versions of these operators, and to what extent the regularity hypotheses in previous articles (often C^∞) can be relaxed.

Endpoint bounds have been established in a number of special cases. A more extensive list of references is given in [5,7]; we will focus here on the most recent results. In [12], Gressman proved that in the polynomial case of the Tao–Wright theorem, endpoint restricted weak type estimates hold, but left open the question of strong type bounds. In the translation-invariant case, more tools are available, and correspondingly, more is known. In [5,13,19], endpoint strong type estimates for convolution with affine arclength measure on polynomial curves were proved. These estimates depend only on the dimension and polynomial degree and require no cutoff function. In [7], an analogous result was proved for the restricted X-ray transform.

For the low regularity case, much less is known. We are primarily motivated by the recent articles [15] and [6]. In [15], Oberlin proved bounds along the sharp line for convolution with affine arclength measure along low-dimensional 'simple' curves satisfying certain monotonicity and log-concavity hypotheses. In particular, there exist infinitely flat curves satisfying these hypotheses. This provides further motivation for the consideration of affine arclength measure, because in these cases there are simply no nontrivial estimates for the unweighted operators. In [6], the first author and Müller proved that restriction to certain C^d perturbations of monomial curves with affine arclength measure satisfies the same range of $L^p \to L^q$ inequalities as restriction to nondegenerate curves.

Our purpose here is to generalize, to the extent possible, the endpoint results mentioned above to more general classes of curves of low regularity. To address the question of the natural relationship between the weighted and unweighted operators, we show how, by a simple interpolation argument, 'weaker' estimates for operators with 'larger' weights (including the optimal estimates in the unweighted case) can be deduced from the optimal estimates for the affine arclength case. Finally, motivated by similarities between restriction operators and generalized Radon transforms, we prove an analogue of the main result of [6] for convolution with affine arclength measure along monomial-like curves with only d derivatives.

2. Results and methods

Let $I \subset \mathbb{R}$ be an interval and $\gamma \in C^d_{\mathrm{loc}}(I; \mathbb{R}^d)$; that is, $\gamma : I \to \mathbb{R}^d$ is a curve in $C^d(K)$ for every compact sub-interval $K \subseteq I$. We define the torsion L_γ and affine arclength measure $\lambda_\gamma\, dt$ by

$$L_\gamma = \det\bigl(\gamma', \ldots, \gamma^{(d)}\bigr), \qquad \lambda_\gamma = |L_\gamma|^{\frac{2}{d(d+1)}}.$$