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Self-adjoint extensions of the Laplace–Beltrami operator and unitaries at the boundary ☆

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ABSTRACT

We construct in this article a class of closed semi-bounded quadratic forms on the space of square integrable functions over a smooth Riemannian manifold with smooth compact boundary. Each of these quadratic forms specifies a semi-bounded self-adjoint extension of the Laplace–Beltrami operator. These quadratic forms are based on the Lagrange boundary form on the manifold and a family of domains parametrized by a suitable class of unitary operators on the boundary that will be called admissible. The corresponding quadratic forms are semi-bounded below and closable. Finally, the representing operators correspond to semi-bounded self-adjoint extensions of the Laplace–Beltrami operator. This family of extensions is compared with results existing in

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Boundary conditions

the literature and various examples and applications are discussed.

Contents

1. Introduction

In this article we construct a family of closed quadratic forms corresponding to a class of self-adjoint extensions of the Laplace–Beltrami operator on a smooth Riemannian manifold with smooth boundary. It is well known that in a smooth manifold Ω with no boundary the minimal closed extension of the Laplace–Beltrami operator $\Delta_{\min}$ is essentially self-adjoint. However, if the manifold has a non-empty boundary $\partial\Omega$, then $\Delta_{\min}$ defines a closed and symmetric but *not self-adjoint* operator. Such situation is common in the study of quantum systems, where some heuristic arguments suggest an expression for the Hamiltonian which is only symmetric. The Laplace–Beltrami operator discussed here can be associated with free quantum systems on the manifold. The description of such systems is not complete until a self-adjoint extension of the Laplace–Beltrami operator has been determined, i.e., a Hamiltonian operator H. Only in this case a unitary evolution of the system is given, because of the one-to-one correspondence between densely defined self-adjoint operators and strongly continuous one-parameter groups of unitary operators $U_t = \exp itH$ provided by Stone's theorem. Therefore the specification of the self-adjoint extension is not just a mathematical artifact, but an essential step in the description of the quantum mechanical system (see, e.g., Chapter X in [40] for further results and motivation).

The collection of all self-adjoint extensions of a densely defined closed symmetric operator T on a complex separable Hilbert space $\mathcal{H}$ was described by von Neumann in terms of the isometries between the deficiency spaces $\mathcal{N}_\pm = \ker(T^\dagger \mp iI)$ of the opera-