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Global maximizers for the sphere adjoint Fourier restriction inequality

Damiano Foschi

Dipartimento di Matematica e Informatica, Università di Ferrara, via Macchiavelli 35, 44121 Ferrara, Italy

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ABSTRACT

We show that constant functions are global maximizers for the adjoint Fourier restriction inequality for the sphere.

1. Introduction

Recently, Christ and Shao [1,2] have proved the existence of maximizers for the adjoint Fourier restriction inequality of Stein and Tomas [5] for the sphere:

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{S}^2)}, \tag{1}$$

where $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is the standard unit sphere equipped with its natural surface measure σ induced by the Lebesgue measure on $\mathbb{R}^3$. Here the Fourier transform

E-mail address: damiano.foschi@unife.it.

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of an integrable function f supported on the sphere is defined for any $x \in \mathbb{R}^3$ by

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^2} \mathrm{e}^{-\mathrm{i}x\cdot\omega} f(\omega)\,\mathrm{d}\sigma_\omega.$$

Let us denote by $\mathcal{R}$ the optimal constant in (1):

$$\mathcal{R} := \sup_{f\in L^2(\mathbb{S}^2), f\neq 0} \frac{\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}}{\|f\|_{L^2(\mathbb{S}^2)}}.$$

In [1], using concentration compactness methods, they prove that there exist sequences $\{f_k\}$ of nonnegative even functions in $L^2(\mathbb{S}^2)$ which converge to some maximizer of the ratio $\|\widehat{f\sigma}\|_{L^4}/\|f\|_{L^2}$, but they do not compute the exact value of $\mathcal{R}$. Nevertheless, they show that constant functions are *local* maximizers and raise the question of whether constants are actually *global* maximizers. The purpose of this note is to give a positive answer to that question:

Theorem 1.1. *A nonnegative function $f \in L^2(\mathbb{S}^2)$ is a global maximizer for (1) if and only if it is a non-zero constant, and we have*

$$\mathcal{R} = \frac{\|\widehat{\mathbf{1}\sigma}\|_{L^4(\mathbb{R}^3)}}{\|\mathbf{1}\|_{L^2(\mathbb{S}^2)}} = 2\pi.$$

When we combine Theorem 1.1 with the results of [2, Theorem 1.2] we obtain that *all* complex-valued global maximizers for (1) are of the form

$$f(\omega) = k\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\xi\cdot\omega},$$

for some $k > 0$, $\theta \in \mathbb{R}$, $\xi \in \mathbb{R}^3$.

A large part of the analysis carried out in [1] is local in nature and it is based on a comparison between the case of the sphere and that of a paraboloid which approximates the sphere at one point. Here we are able to keep everything global, thanks to an interesting geometric feature of the sphere, which is expressed in Lemma 4.2. It essentially says: when the sum $\omega_1 + \omega_2 + \omega_3$ of three unit vectors is again a unit vector, then we have

$$|\omega_1 + \omega_2|^2 + |\omega_1 + \omega_3|^2 + |\omega_2 + \omega_3|^2 = 4.$$

In order to find maximizers for (1), we follow the spirit of the proof of analogous results obtained by the author for the paraboloid and the cone [4]. The main steps are:

- The exponent 4 is an even integer and we can view the L^4 norm as an L^2 norm of a product, which becomes, through the Fourier transform, an L^2 norm of a convolution. We write the L^2 norm of a convolution of measures supported on the sphere as a quadrilinear integral over a submanifold of $(\mathbb{S}^2)^4$.