Journal of Functional Analysis 268 (2015) 732–747

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Anderson localization for the completely resonant phases ☆

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ARTICLE INFO

Article history:
Received 13 November 2013
Accepted 28 October 2014
Available online 17 November 2014
Communicated by L. Gross

Keywords:
Anderson localization
Almost Mathieu operator
Resonant phases

ABSTRACT

For the almost Mathieu operator $(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + 2\lambda\cos 2\pi(\theta + n\alpha)u(n)$, Avila and Jitomirskaya conjecture that for every phase $\theta \in \mathscr{R} \triangleq \{\theta \in \mathbb{R} \mid 2\theta + \alpha\mathbb{Z} \in \mathbb{Z}\}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta}$. In the present paper, we verify the conjecture for $|\lambda| > e^{7\beta}$.

1. Introduction

The almost Mathieu operator (AMO) is the quasi-periodic Schrödinger operator on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)(n) = u(n+1) + u(n-1) + \lambda v(\theta + n\alpha)u(n), \quad \text{with } v(\theta) = 2\cos 2\pi\theta, \tag{1.1}$$

where λ is the coupling, α is the frequency, and θ is the phase.

AMO is the most studied quasi-periodic Schrödinger operator, arising naturally as a physical model (see [9] for a recent historical account and the physics background).

☆ Supported by NSFC11271076 and NSFC11121101.
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http://dx.doi.org/10.1016/j.jfa.2014.10.024
0022-1236/

We say a phase $\theta \in \mathbb{R}$ is completely resonant with respect to frequency α, if $\theta \in \mathscr{R} \triangleq \{\theta \in \mathbb{R} \mid 2\theta + \alpha\mathbb{Z} \in \mathbb{Z}\}$.

Anderson localization (i.e., only pure point spectrum with exponentially decaying eigenfunctions) is not only meaningful in physics, but also related to reducibility for Aubry dual model (see [6]). For example, Puig [12,13] shows that Anderson localization for completely resonant phases will lead to reducibility to the parabolic cocycles, which may become uniform hyperbolic cocycles under small perturbation of energies. Furthermore, Anderson localization for completely resonant phases is crucial in describing open gaps of $\Sigma_{\lambda,\alpha}$ (the spectrum of $H_{\lambda,\alpha,\theta}$ is independent of θ for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, denoted by $\Sigma_{\lambda,\alpha}$). See [2,11] for details.

It is well known that $H_{\lambda,\alpha,\theta}$ has purely absolutely continuous spectrum for $\alpha \in \mathbb{Q}$ and all λ. This implies $H_{\lambda,\alpha,\theta}$ can not satisfy Anderson localization for all $\alpha \in \mathbb{Q}$. Thus we always assume $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ in the present paper.

The following notions are useful in the study of Eq. (1.1).

We say $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ satisfies Diophantine condition $\mathrm{DC}(\kappa, \tau)$ with $\kappa > 0$ and $\tau > 0$, if

$$\|k\alpha\|_{\mathbb{R}/\mathbb{Z}} > \kappa |k|^{-\tau} \quad \text{for any } k \in \mathbb{Z} \setminus \{0\},$$

where $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{\ell \in \mathbb{Z}} |x - \ell|$. Let $\mathrm{DC} = \bigcup_{\kappa>0,\, \tau>0} \mathrm{DC}(\kappa, \tau)$. We say α satisfies Diophantine condition, if $\alpha \in \mathrm{DC}$.

Let

$$\beta = \beta(\alpha) := \limsup_{n\to\infty} \frac{\ln q_{n+1}}{q_n}, \tag{1.2}$$

where $\frac{p_n}{q_n}$ is the continued fraction approximants to α. Notice that $\beta(\alpha) = 0$ for $\alpha \in \mathrm{DC}$.

Avila and Jitomirskaya conjecture that for any completely resonant phase θ, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $|\lambda| > e^{2\beta}$ [1, Remark 9.1]. Jitomirskaya, Koslover and Schulteis [8] prove this for $\alpha \in \mathrm{DC}$ via a simple modification of the proof in [7]. More concretely, for $\alpha \in \mathrm{DC}$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathscr{R}$ and $|\lambda| > 1$. In [2], Avila and Jitomirskaya develop a quantitative version of Aubry duality. Then, they obtain that for α with $\beta(\alpha) = 0$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathscr{R}$ and $|\lambda| > 1$. The present authors extend the quantitative version of Aubry duality to all α with $\beta(\alpha) < \infty$, and show that for all α with $\beta(\alpha) < \infty$, $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathscr{R}$ and $|\lambda| > e^{C\beta}$, where C is a large absolute constant [11]. In the present paper, we give a definite quantitative description about the constant C, and obtain the following theorem.

Theorem 1.1. *For $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\beta(\alpha) < \infty$, the almost Mathieu operator $H_{\lambda,\alpha,\theta}$ satisfies Anderson localization if $\theta \in \mathscr{R}$ and $|\lambda| > e^{7\beta}$, where $\mathscr{R} \triangleq \{\theta \in \mathbb{R} \mid 2\theta + \alpha\mathbb{Z} \in \mathbb{Z}\}$.*

Remark 1.1. Avila and Jitomirskaya think that $H_{\lambda,\alpha,0}$ does not display Anderson localization if $|\lambda| \le e^{2\beta}$ [1, Remark 5.2]. This is still open. Clearly, $0 \in \mathscr{R}$.