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Semicircular limits on the free Poisson chaos: Counterexamples to a transfer principle

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We establish a class of sufficient conditions ensuring that a sequence of multiple integrals with respect to a free Poisson measure converges to a semicircular limit. We use this result to construct a set of explicit counterexamples showing that the *transfer principle* between classical and free Brownian motions (recently proved by Kemp, Nourdin, Peccati and Speicher (2012)) does not extend to the framework of Poisson measures. Our counterexamples implicitly use kernels appearing in the classical theory of random geometric graphs. Several new results of independent interest are obtained as necessary steps in our analysis, in particular: (i) a multiplication formula for free Poisson multiple integrals, (ii) diagram formulae and spectral bounds for these objects, and (iii) a counterexample to the general universality of the Gaussian Wiener chaos in a classical setting.

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1. Introduction

Let $W = \{W_t : t \geq 0\}$ be a real-valued standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $S = \{S_t : t \geq 0\}$ be a free Brownian motion defined on the non-commutative probability space $(\mathscr{A}, \varphi)$. Given $q \geq 1$ and a kernel $f \in L^2(\mathbb{R}_+^q, \mu^q)$ (where μ stands for the Lebesgue measure), we shall denote by $I_q^S(f)$ and $I_q^W(f)$, respectively, the *multiple Wigner integral* of f with respect to S (see [10]), and the *multiple Wiener integral* of f with respect to W (see [22]). Objects of this type play crucial roles in non-commutative and classical stochastic analysis; in particular, they constitute the basic building blocks that allow one to define Malliavin operators in both contexts (see again [10,22]).

The following asymptotic result, that contains both a semicircular limit theorem and a transfer principle, was first proved in [16]. We denote by $\mathcal{S}(0,1)$ (respectively $\mathcal{N}(0,1)$) a centered semicircular random variable (respectively centered Gaussian random variable) with unit variance (all the notions evoked in the introduction will be formally defined in Section 2 and Section 3).

Theorem 1.1. *(See [16].) Fix an integer $q \geq 2$, and let $\{f_n : n \geq 1\} \subset L^2(\mathbb{R}_+^q)$ be a sequence of mirror symmetric (cf. Notation 3.1 below for a definition of mirror symmetry) kernels such that $\|f_n\|_{L^2(\mathbb{R}_+^q)} \to 1$.*

(A) *As $n \to \infty$, one has that $I_q^S(f_n)$ converges in law to $\mathcal{S}(0,1)$ if and only if $\varphi(I_q^S(f_n)^4) \to \varphi(\mathcal{S}(0,1)^4) = 2$.*
(B) (Transfer principle) *If each f_n is fully symmetric (cf. again Notation 3.1 for a definition of full symmetry), then $I_q^S(f_n)$ converges in law to $\mathcal{S}(0,1)$ if and only if $I_q^W(f_n)$ converges in law to $\sqrt{q!}\mathcal{N}(0,1)$.*

Part (A) of the previous statement is indeed a free counterpart of a *fourth moment central limit theorem* by Nualart and Peccati, originally proved in the framework of multiple Wiener integrals with respect to a Gaussian field (see e.g. [21,24], as well as [22, Chapter 5]). On the other hand, Part (B) establishes a transfer principle according to which, in the presence of symmetric kernels, convergence to the semicircular distribution on the Wigner chaos is exactly equivalent to a chaotic Gaussian limit on the classical Wiener space. Recall that, in free probability, the semicircular distribution plays the same central role as the Gaussian distribution in the classical context. As shown in [13,16], such a transfer principle allows one to deduce almost immediately free versions of important central limit theorems from classical stochastic analysis, like for instance non-commutative versions of the famous Breuer–Major theorem for Gaussian-subordinated sequences (see [22, Chapter 7]).

Since its appearance, Theorem 1.1 has triggered a number of generalizations: see e.g. [26] for multidimensional statements, [23] for convergence results towards the free Poisson distribution, [20] for alternate proofs, [6] for fourth moment theorems in the