Journal of Functional Analysis 267 (2014) 998–1010

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Cases of equality in certain multilinear inequalities of Hardy–Riesz–Rogers–Brascamp–Lieb–Luttinger type ☆

Michael Christ *, Taryn C. Flock

Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

A R T I C L E   I N F O

Article history:
Received 27 December 2013
Accepted 11 March 2014
Available online 4 June 2014
Communicated by F. Barthe

Keywords:
Multilinear inequalities
Rearrangement inequalities

A B S T R A C T

Cases of equality in certain Hardy–Riesz–Rogers–Brascamp–Lieb–Luttinger rearrangement inequalities are characterized.

1. Statement of result

Let $m \geq 2$ and $n \geq m+1$ be positive integers. For $j \in \{1, 2, \cdots, n\}$ let $E_j \subset \mathbb{R}$ be Lebesgue measurable sets with positive, finite measures, and let L_j be surjective linear maps $\mathbb{R}^m \to \mathbb{R}$. This paper concerns the nature of those n-tuples $(E_1, \cdots, E_n)$ of measurable sets that maximize expressions

$$I(E_1, \cdots, E_n) = \int_{\mathbb{R}^m} \prod_{j=1}^{n} \mathbb{1}_{E_j}\big(L_j(x)\big)\, dx,$$

☆ The authors were supported in part by NSF grant DMS-0901569.

* Corresponding author.
E-mail addresses: mchrist@berkeley.edu (M. Christ), taryn.flock@aya.yale.edu (T.C. Flock).

http://dx.doi.org/10.1016/j.jfa.2014.03.005
0022-1236/© 2014 Published by Elsevier Inc.

among all n-tuples with specified Lebesgue measures $|E_j|$. Our results apply only in the lowest-dimensional nontrivial case, $m = 2$, but apply for arbitrarily large n.

Definition 1. A family $\{L_j\}$ of surjective linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^1$ is nondegenerate if for every set $S \subset \{1, 2, \cdots, n\}$ of cardinality m, the map $x \mapsto (L_j(x) : j \in S)$ from $\mathbb{R}^m$ to $\mathbb{R}^S$ is a bijection.

For any Lebesgue measurable set $E \subset \mathbb{R}^1$ with finite Lebesgue measure, E^* denotes the nonempty closed[1] interval centered at the origin satisfying $|E| = |E^*|$. Rogers [9] and Brascamp, Lieb, and Luttinger [1] proved that among sets with specified measures, the functional I attains its maximum value when each E_j equals E_j^*, that is,

$$I(E_1, \cdots, E_n) \le I\big(E_1^*, \cdots, E_n^*\big). \tag{1}$$

In this paper we study the uniqueness question and show that these are the only maximizing n-tuples, up to certain explicit symmetries of the functional, in those situations in which a satisfactory characterization of maximizers can exist.

Inequalities of this type can be traced back at least to Hardy and to Riesz [8]. In the 1930s, Riesz and Sobolev independently showed that

$$\iint\limits_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(y)\mathbb{1}_{E_3}(x+y)\,dx\,dy \le \iint\limits_{\mathbb{R}^k \times \mathbb{R}^k} \mathbb{1}_{E_1^*}(x)\mathbb{1}_{E_2^*}(y)\mathbb{1}_{E_3^*}(x+y)\,dx\,dy$$

for arbitrary measurable sets E_j with finite Lebesgue measures. Later Rogers [9] and Brascamp, Lieb, and Luttinger [1] proved the more general result indicated above, and in a yet more general form in which the target spaces $\mathbb{R}^1$ are replaced by $\mathbb{R}^k$ for arbitrary $k \ge 1$, satisfying an appropriate equivariance hypothesis.

The first inverse theorem in this context, characterizing cases of equality, was established by Burchard [3,2]. The cases $n \le m$ are uninteresting, since $I(E_1, \cdots, E_n) = \infty$ for all $(E_1, \cdots, E_n)$ when $n < m$, and equality holds for all sets when $n = m$. The results of Burchard [2] apply to the smallest nontrivial value of n for given m, that is to $n = m+1$, but not to larger n. We are aware of no further progress in this direction since that time. This paper treats a situation at the opposite extreme of the spectrum of possibilities, in which $m = 2$ is the smallest dimension of interest, but the number $n \ge 3$ of factors can be arbitrarily large.

Burchard's inverse theorem has more recently been applied to characterizations of cases of equality in certain inequalities for the Radon transform and its generalizations the k-plane transforms [4,7]. Cases of near but not exact equality for the Riesz–Sobolev inequality have been characterized still more recently [5,6].

[1] A more common convention is that E^* should be open, but this convention will be convenient in our proofs. If $E = \emptyset$ then $E^* = \{0\}$, rather than the empty set, under our convention.