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Operator algebras and subproduct systems arising from stochastic matrices ☆

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ABSTRACT

We study subproduct systems in the sense of Shalit and Solel arising from stochastic matrices on countable state spaces, and their associated operator algebras. We focus on the non-self-adjoint tensor algebra, and Viselter's generalization of the Cuntz–Pimsner C*-algebra to the context of subproduct systems. Suppose that X and Y are Arveson–Stinespring subproduct systems associated to two stochastic matrices over a countable set Ω, and let $\mathcal{T}_+(X)$ and $\mathcal{T}_+(Y)$ be their tensor algebras. We show that every algebraic isomorphism from $\mathcal{T}_+(X)$ onto $\mathcal{T}_+(Y)$ is automatically bounded. Furthermore, $\mathcal{T}_+(X)$ and $\mathcal{T}_+(Y)$ are isometrically isomorphic if and only if X and Y are unitarily isomorphic up to a *-automorphism of $\ell^\infty(\Omega)$. When Ω is finite, we prove that $\mathcal{T}_+(X)$ and $\mathcal{T}_+(Y)$ are algebraically isomorphic if and only if there exists a similarity between X and Y up to a *-automorphism of $\ell^\infty(\Omega)$. Moreover, we provide an explicit description of the Cuntz–Pimsner algebra $\mathcal{O}(X)$ in the case where Ω is finite and the stochastic matrix is essential.

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1. Introduction

In this paper we study the structure of tensor and Cuntz–Pimsner algebras (in the sense of Viselter [35]) associated to subproduct systems, and to what extent these algebras provide invariants for their subproduct systems. These algebras generalize the tensor and Cuntz–Pimsner operator algebras associated to C*-correspondences, which have been the focus of considerable interest by many researchers. Tensor algebras of a C*-correspondence, in particular, have been the subject of a deep study by Muhly and Solel [21–23], which has led into a far-reaching non-commutative generalization of function theory. We will focus on subproduct systems associated to stochastic matrices, and in this context we prove several results which have a close parallel in the work of Davidson, Ramsey and Shalit [10] on the isomorphism problem of tensor algebras of subproduct systems over $\mathbb{C}$ with finite dimensional (Hilbert space) fibers.

A subproduct system over a W*-algebra $\mathcal{M}$ (and over the additive semigroup $\mathbb{N}$) is a family $\{X_n\}_{n\in\mathbb{N}}$ of W*-correspondences over $\mathcal{M}$ endowed with an isometric comultiplication $X_{n+m} \to X_n \otimes X_m$ which is an adjointable $\mathcal{N}$-bimodule map for every n, m. Subproduct systems were first defined and studied for their own sake by Shalit and Solel [28], and in the special case of $\mathcal{M} = \mathbb{C}$ they were also independently studied under the name of inclusion systems by Bhat and Mukherjee [4]. Subproduct systems had appeared implicitly earlier in the work of many researchers in the study of dilations of semigroups of completely positive maps (cp-semigroups for short) on von Neumann algebras and later C*-algebras (see for example [5,23,20]). The study of cp-semigroups is closely related to the analysis of E_0-semigroups and product systems pioneered by Arveson and Powers (for a comprehensive introduction see [2], and also [32] for product systems of Hilbert modules).

Given a correspondence E over a C*-algebra $\mathcal{A}$, the Toeplitz C*-algebra $\mathcal{T}(E)$ and the Cuntz–Pimsner C*-algebra $\mathcal{O}(E)$ were introduced by Pimsner [26], and modified by Katsura [17] in the case of non-injective left action of $\mathcal{A}$. As is well-known, in general the Cuntz–Pimsner algebra does not provide a very strong invariant of the underlying correspondence. However, some information does remain. In the case of graph C*-algebras, for example, if a graph is row-finite, then its C*-algebra is simple if and only if the graph is cofinal and every cycle has an entry. And it is easy to find two graphs with d vertices and irreducible adjacency matrix whose C*-algebras are not isomorphic (see [27]). In contrast, in Section 5 we show that if X is the Arveson–Stinespring subproduct system of a $d \times d$ irreducible stochastic matrix, then $\mathcal{O}(X) \cong C(\mathbb{T}) \otimes M_d(\mathbb{C})$. More generally, we also provide an explicit description for the Cuntz–Pimsner algebra of a subproduct system associated to essential finite stochastic matrices.

On the other hand, the non-self-adjoint tensor algebra $\mathcal{T}_+(E)$ of a C*-correspondence E over $\mathcal{A}$ has often proven to be a strong invariant of the correspondence. Muhly and Solel [22] proved that if E and F are aperiodic C*-correspondences, then $\mathcal{T}_+(E)$ is isometrically isomorphic to $\mathcal{T}_+(F)$ if and only if E and F are isometrically isomorphic as $\mathcal{A}$-bimodules. Similarly, Katsoulis and Kribs [16] and Solel [33] proved