Journal of Functional Analysis 267 (2014) 1121–1136

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A noncommutative Amir–Cambern theorem for von Neumann algebras and nuclear C^*-algebras

Éric Ricard [a], Jean Roydor [b,*]

[a] *Laboratoire de Mathématiques Nicolas Oresme, Université de Caen Basse-Normandie, 14032 Caen Cedex, France*

[b] *Institut de Mathématiques de Bordeaux, Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence Cedex, France*

ARTICLE INFO

Article history:
Received 21 February 2014
Accepted 23 May 2014
Available online 5 June 2014
Communicated by S. Vaes

Keywords:
Completely bounded maps
Haagerup tensor product
(Completely bounded) Hochschild cohomology
Length of an operator algebra

ABSTRACT

We prove that von Neumann algebras and separable nuclear C^*-algebras are stable for the Banach–Mazur cb-distance. A technical step is to show that unital almost completely isometric maps between C^*-algebras are almost multiplicative and almost selfadjoint. Also as an intermediate result, we compare the Banach–Mazur cb-distance and the Kadison–Kastler distance. Finally, we show that if two C^*-algebras are close enough for the cb-distance, then they have comparable length.

1. Introduction

This paper concerns perturbations of operator algebras as operator spaces, more precisely perturbations relative to the Banach–Mazur cb-distance. In [19], G. Pisier introduced the Banach–Mazur cb-distance (or cb-distance in short) between two operator spaces $\mathcal{X}$, $\mathcal{Y}$:

$$d_{cb}(\mathcal{X},\mathcal{Y}) = \inf\{\|T\|_{cb}\|T^{-1}\|_{cb}\},$$

* Corresponding author.
E-mail addresses: eric.ricard@unicaen.fr (É. Ricard), jean.roydor@math.u-bordeaux1.fr (J. Roydor).

http://dx.doi.org/10.1016/j.jfa.2014.05.018

where the infimum runs over all possible linear completely bounded isomorphisms $T: \mathcal{X} \to \mathcal{Y}$. This extends naturally the classical Banach–Mazur distance of Banach spaces when these are endowed with their minimal operator space structure (in particular, the Banach–Mazur distance and the cb-distance between two $C(K)$-spaces coincide). For background on completely bounded maps and operator space theory the reader is referred to [2,11,18] and [22].

For $C(K)$-spaces, being isomorphic (or equivalently cb-isomorphic) is a very flexible relation; Milutin's theorem (see [15] Theorem 2.1) states that $C(K)$ is isomorphic to $C([0,1])$ for any uncountable compact metric space K. This theorem has some noncommutative generalization for separably acting injective von Neumann algebras (see [7]) and for separable non-type I nuclear C^*-algebras (see [17]). But here, we are interested in perturbation results when the cb-distance is small. Let us recall the generalization of Banach–Stone theorem obtained independently by D. Amir and M. Cambern (see [1,3]): if the Banach–Mazur distance between two $C(K)$-spaces is strictly smaller than 2, then they are $*$-isomorphic (as C^*-algebras). Actually, this is also true for spaces of continuous functions vanishing at infinity on locally compact Hausdorff spaces. One is tempted to extend the Amir–Cambern theorem to noncommutative C^*-algebras (see [5]). In [16], R. Kadison described isometries between C^*-algebras, in particular the isometric structure of a C^*-algebra determines its Jordan structure. Moreover, A. Connes (see [10]) exhibited a factor not $*$-isomorphic to its opposite factor (but obviously a C^*-algebra and its opposite algebra are always isometric). Therefore, to recover the C^*-structure, we need assumption on the cb-distance (not only on the classical Banach–Mazur distance). Here, we prove:

Theorem A. *Let $\mathcal{A}$ be a separable nuclear C^*-algebra or a von Neumann algebra, then there exists an $\varepsilon_0 > 0$ such that for any C^*-algebra $\mathcal{B}$, the inequality $d_{cb}(\mathcal{A}, \mathcal{B}) < 1 + \varepsilon_0$ implies that $\mathcal{A}$ and $\mathcal{B}$ are $*$-isomorphic.*

When $\mathcal{A}$ is a separable nuclear C^*-algebra, one can take $\varepsilon_0 = 3.10^{-19}$. When $\mathcal{A}$ is a von Neumann algebra, $\varepsilon_0 = 4.10^{-6}$ is sufficient.

Such a result cannot be extended to all unital C^*-algebras, see Corollary 3.9 below for a counter-example (derived from [4]) involving nonseparable C^*-algebras.

The proof of Theorem A is totally different from the commutative case, the cb-distance concerns only the operator space structure, hence the basic idea is to gain the algebraic structure. It is known that unital completely isometric linear isomorphisms between operator algebras are necessarily multiplicative (see Theorem 4.5.13 [2]). Therefore, one wants to prove that unital almost completely isometric maps are almost multiplicative; in the sense that the defect of multiplicativity has small cb-norm as a bilinear map. We manage to check this by an ultraproduct argument (see Proposition 2.1) but without any explicit control on the defect of multiplicativity. When maps are between C^*-algebras, we can drop the 'unital' hypothesis and show that the unitization of an almost completely isometric map between C^*-algebras is almost multiplicative with explicit