Journal of Functional Analysis 267 (2014) 1256–1284

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Long time existence of smooth solution for the porous medium equation in a bounded domain

Sunghoon Kim

Department of Mathematics, School of Natural Sciences, The Catholic University of Korea, 43 Jibong-ro, Wonmi-gu, Bucheon-si, Gyeonggi-do, 420-743, Republic of Korea

ARTICLE INFO

Article history:
Received 5 April 2014
Accepted 18 May 2014
Available online 2 June 2014
Communicated by S. Brendle

Keywords:
Porous medium equation in a bounded domain
Hölder estimate
Long time existence

ABSTRACT

In this paper, we are going to show the long time existence of the smooth solution for the porous medium equations in a smooth bounded domain:

$$\begin{cases} u_t = \triangle u^m & \text{in } \Omega \times [0,\infty), \\ u(x,0) = u_0 > 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \end{cases} \tag{0.1}$$

where $m > 1$ is the permeability. The proof is based on the short time existence of $C_s^{2,\bar{\gamma}}$-smooth solution, the global C_s^1-estimate, the Hölder estimate of divergence type degenerate equation with measurable coefficients and $C_s^{1,\bar{\gamma}}$-estimate of mixed type equation with Lipschitz coefficients.

1. Introduction

We consider in this paper the initial value problem for the *Porous Medium Equation* (PME)

E-mail address: math.s.kim@catholic.ac.kr.

http://dx.doi.org/10.1016/j.jfa.2014.05.017

$$\begin{cases} u_t = \triangle u^m & \text{in } \Omega, \\ u(x,0) = u_0 > 0 & \text{in } \Omega, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega \end{cases} \tag{1.1}$$

posed in a bounded domain Ω with the range of exponents $m > 1$, with initial data u_0 nonnegative, integrable and compactly supported.

In the case of $\Omega = \mathbb{R}^n$, it becomes a free boundary problem where the region $\{u > 0\}$ propagates with a finite speed. It is well known that if the initial data u_0 is nonnegative, integrable and compactly supported, then the Cauchy problem for the porous medium equation admits a unique solution on $\mathbb{R}^n \times (0, \infty)$ which has constant mass. By the phenomenon of the finite speed propagation, the solution $u(\cdot, t)$ will remain compactly supported for all time $t > 0$ when the initial date u_0 is compactly supported in $\mathbb{R}^n$. Caffarelli and Friedman [4,5] showed that the interface can always be described by a Hölder continuous function $t = S(x)$, $x \notin \text{supp}(u_0)$, for any initial data. The regularity of the free boundary has been studied by many authors. Caffarelli and Wolanski [7] showed that, under the assumption that at time $t = 0$ the pressure $v_0 = mu_0^{m-1} \in C^1(\overline{\text{supp}(v_0)})$ and $D(v_0) \neq 0$ along $\partial\{\overline{\text{supp}(v_0)}\}$, a condition which ensures that the free boundary will start to move at each point at $t = 0$, the free boundary is a $C^{1,\beta}$ surface. Daskalopoulos and Hamilton [8] showed that under rather general assumptions on the initial data, the free boundary is smooth surface when $0 < t < T$, for some $T > 0$.

In the case of bounded domain Ω, Vázquez [18,17] showed that the nonnegative weak solutions exist, are unique and depend continuously on the initial data in the $L^1(\Omega)$ norm. The concept of viscosity solutions for the PME was established by Caffarelli and Vázquez [6]. In [3], the authors extend the idea of viscosity solutions to quasilinear equations of the form $u_t = a(u)\triangle u + |\nabla u|^2$ where the coefficient a vanishes at $u = 0$. The large-time stabilization for the solutions of (1.1) has been studied by Aronson and Peletier [2], who proved that as $t \to \infty$, the solutions tend in the L^∞ norm to the similarity solution $U(x,t) = \frac{g(x)}{(1+t)^{\frac{1}{m-1}}}$ with an error of order $O(\frac{1}{(1+t)^{\frac{m}{m-1}}})$. Here g is the unique solution of the elliptic equation

$$\triangle g^m + \frac{g}{m-1} = 0$$

with $g > 0$ in Ω and $g = 0$ on $\partial\Omega$. To get the desired result, they proved a comparison theorem for weak solutions. Lee and Vázquez [16] improved the rate of convergence to the limit profile g and provided parabolic method for describing the geometric properties of the elliptic nonlinear eigenvalue problem satisfied by g^m. The convergence rate of the solutions with compact support in $\mathbb{R}^n$ has been discussed for the porous medium equations [15], and for the parabolic p-Laplacian type [14].

In [11], Kim and Lee dealt with the short time existence of solution to PME in a bounded domain. More precisely, the main result in their paper is that, if u is a solution to (1.1) and $f = u^m$, then, in some regularity conditions on the initial data $f^0 = u_0^m$ and its first and second derivatives, the solution of the degenerate equation