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Toeplitz determinants with perturbations in the corners [☆]

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The paper is devoted to exact and asymptotic formulas for the determinants of Toeplitz matrices with perturbations by blocks of fixed size in the four corners. If the norms of the inverses of the unperturbed matrices remain bounded as the matrix dimension goes to infinity, then standard perturbation theory yields asymptotic expressions for the perturbed determinants. This premise is not satisfied for matrices generated by so-called Fisher–Hartwig symbols. In that case we establish formulas for pure single Fisher–Hartwig singularities and for Hermitian matrices induced by general Fisher–Hartwig symbols.

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1. Introduction

This paper was prompted by a problem from lattices associated with finite Abelian groups. This problem, which will be described in Section 2, led to the computation of the determinant of the $n \times n$ analogue A_n of the matrix

$$A_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 1 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 1 & 0 & 0 & 1 & -4 & 6 \end{pmatrix}. \quad (1)$$

It turns out that $\det A_n = (n+1)^3$. What makes the matter captivating is that the determinant of the $n \times n$ version T_n of

$$T_6 = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 1 & -4 & 6 \end{pmatrix} \quad (2)$$

is a so-called pure Fisher–Hartwig determinant. The latter determinant is known to be

$$\frac{(n+1)(n+2)^2(n+3)}{12}. \quad (3)$$

This formula was established in [3]. See also [5, Theorem 10.59] or [6]. We were intrigued by the question why the perturbations in the corners lower the growth from n^4 to n^3.

The general context is as follows. Every complex-valued function $a \in L^1$ on the unit circle $\mathbf{T}$ has well-defined Fourier coefficients

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(e^{i\theta}) e^{-ik\theta}\, d\theta, \quad k \in \mathbf{Z},$$

and generates the infinite Toeplitz matrix $T(a) = (a_{j-k})_{j,k=1}^{\infty}$. The principal $n \times n$ truncation of this matrix is denoted by $T_n(a)$. Thus, $T_n(a) = (a_{j-k})_{j,k=1}^{n}$. The function a is usually referred to as the symbol of the infinite matrix $T(a)$ and of the sequence $\{T_n(a)\}_{n=1}^{\infty}$. For example, matrix (2) is just $T_6(a)$ with

$$a(t) = t^{-2} - 4t^{-1} + 6 - 4t + t^2 = \left(1 - \frac{1}{t}\right)^2 (1-t)^2 = |1-t|^4, \quad (4)$$