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Essential normality of polynomial-generated submodules: Hardy space and beyond

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Abstract

Recently, Douglas and Wang proved that for each polynomial q, the submodule $[q]$ of the Bergman module generated by q is essentially normal [11]. Using improved techniques, we show that the Hardy space analogue of this result holds, and more.

Keywords: Essential normality; Submodule; Polynomial

1. Introduction

Let $\mathbf{B}$ be the unit ball in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury–Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ with $(1 - \langle \zeta, z\rangle)^{-1}$ as its reproducing kernel. The space H_n^2 is naturally considered as a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$. In [3–6], Arveson raised the question of whether graded submodules $\mathcal{M}$ of H_n^2 are essentially normal. That is, for the restricted operators

$$Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \quad 1 \leqslant j \leqslant n,$$

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on $\mathcal{M}$, do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,9,12,15,16]. In particular, Guo and Wang showed that the answer to the above question is affirmative if $\mathcal{M}$ is generated by a homogeneous polynomial [16]. In [10], Douglas proposed analogous essential normality problems for submodules of the Bergman module $L^2_a(\mathbf{B}, dv)$.

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [11] for *arbitrary* polynomials. In that paper, Douglas and Wang showed that for any polynomial $q \in \mathbf{C}[z_1, \dots, z_n]$, the submodule $[q]$ of the Bergman module generated by q is p-essentially normal for $p > n$. What is especially remarkable is that [11] contains many novel ideas.

The present paper grew out of a remark in [11]. Toward the end of [11], Douglas and Wang commented

"*It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury–Arveson spaces. However, while we believe that both results hold, perhaps techniques from* [9,7] *may be needed to complete the proofs*."

In the above direct quote, [9] and [7] are the original reference numbers in the Douglas–Wang paper. In the present paper, these references are cited as [8] and [7] respectively.

In this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

The key realization is that Bergman space, Hardy space and Drury–Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on $\mathbf{B}$ parametrized by a real-valued parameter $-n \leqslant t < \infty$. In fact, the spaces corresponding to the values $t \in \mathbf{Z}_+$ were used in an essential way in the proofs in [11]. Our main observation is that if one considers other values of t, then one will see how to extend the techniques in [11] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [11] for spaces with parameter $-2 < t < \infty$. Before stating the result, let us first introduce these spaces.

For each real number $-n \leqslant t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \dots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}^n_+ \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Here and throughout the paper, we use the conventional multi-index notation [17, p. 3].