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Globally well and ill posedness for non-elliptic derivative Schrödinger equations with small rough data

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Abstract

We show that there exists $s_c > 0$ such that the cubic (quartic) non-elliptic derivative Schrödinger equations with small data in modulation spaces $M^s_{2,1}(\mathbb{R}^n)$ for $n \geqslant 3$ ($n = 2$) are globally well-posed if $s \geqslant s_c$, and ill-posed if $s < s_c$. In 2D cubic case, using the Gabor frame, we get some time-global dispersive estimates for the Schrödinger semi-group in anisotropic Lebesgue spaces, which include a time-global maximal function estimate in the space $L^2_{x_1} L^\infty_{x_2,t}$. By resorting to the smooth effect estimate together with the dispersive estimates in anisotropic Lebesgue spaces, we show that the cubic hyperbolic derivative NLS in 2D has a unique global solution if the initial data in Feichtinger–Segal algebra or in weighted Sobolev spaces are sufficiently small.

Keywords: Non-elliptic derivative Schrödinger equation; Gabor frame; Modulation spaces; Well posedness; Ill posedness

1. Introduction

In this paper we consider the Cauchy problem for the derivative nonlinear Schrödinger equation (DNLS):

$$\mathrm{i}u_t - \Delta_\pm u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x), \tag{1.1}$$

where $\Delta_\pm = \varepsilon_1 \partial^2_{x_1} + \cdots + \varepsilon_n \partial^2_{x_n}$, $\varepsilon_i \in \{1, -1\}$ for $i = 1, \ldots, n$, $\nabla u = (u_{x_1}, \ldots, u_{x_n})$,

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$$F(z)=F(z_1,\dots,z_{2n+2})=\sum_{m+1\leqslant|\beta|<\infty} c_\beta z^\beta,\quad m\geqslant 2,\ c_\beta\in\mathbb{C} \tag{1.2}$$

and $|c_\beta|\leqslant C^{|\beta|}$ for $\beta=(\beta_1,\dots,\beta_{2n+2})$. A special case of (1.1) is the following

$$\mathrm{i}u_t-\Delta_\pm u=\vec{\lambda}\cdot\nabla\big(|u|^{2\kappa}u\big)+\mu|u|^{2\nu}u,\quad u(0,x)=u_0(x), \tag{1.3}$$

$\vec{\lambda}\in\mathbb{C}^n$, $\mu\in\mathbb{C}$ and $\kappa,\nu\in\mathbb{N}$. It is known that (1.3) ($\kappa=1$) arises from the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics [6,10,34], where anisotropic interactions are manifested by the presence of the non-elliptic case, as well as additional residual terms which involve cross derivatives of the independent variables. Another typical example is the Schrödinger map equation

$$\mathrm{i}u_t+\Delta_\pm u=\frac{2\bar{u}}{1+|u|^2}\big(\varepsilon_1 u_{x_1}^2+\dots+\varepsilon_n u_{x_n}^2\big),\quad u(0,x)=u_0(x), \tag{1.4}$$

which is an equivalent form of the non-elliptic Schrödinger map

$$s_t=s\times\Delta_\pm s,\quad s(0,x)=s_0(x). \tag{1.5}$$

where $s=(s_1,s_2,s_3)$, $s:\mathbb{R}\times\mathbb{R}^n\to\mathbb{S}^2$ is a real valued map of $(t,x_1,\dots,x_n)$. Indeed, if u satisfies (1.4), taking

$$s=\left(\frac{2\,Re\,u}{1+|u|^2},\frac{2\,Im\,u}{1+|u|^2},\frac{1-|u|^2}{1+|u|^2}\right),$$

we see that s is the solution of (1.5). Conversely, taking u as the stereographic projection of s defined by

$$u=\frac{s_1+is_2}{1+s_3},$$

we see that (1.5) reduces to (1.4). A large amount of work has been devoted to the study of the elliptic Schrödinger map initial value problem ($\Delta_\pm=\Delta$) together with their generalizations [1,9,24,29,40,39,41].

If the initial data have no decay on $x\in\mathbb{R}^n$, say only in C^∞, it is easy to give a finite-time blow up solution of the derivative NLS in 2D. In the non-elliptic case, all of the blow up points can constitute infinitely many hyperbolic curves, see Appendix A.

For the general equation (1.1) in the elliptic case with nonlinearity (1.2), the local and global well posedness were studied in [5,17,21,22,25–27]. When the nonlinear term F satisfies an energy structure condition $Re\,\partial F/\partial(\nabla u)=0$ and the initial data are sufficiently smooth in weighted Sobolev spaces, Klainerman [21], Shatah [27] and Klainerman and Ponce [22] obtained the global existence of (1.1) in all spatial dimensions. Chihara [5] considered the initial data in sufficiently smooth weighted Sobolev spaces and removed the energy structure condition $Re\,\partial F/\partial(\nabla u)=0$ for $n\geqslant 3$ and only assume that cubic terms $F_1(z)$ in $F(z)$ is modulation homogeneous (i.e., $F_1(e^{\mathrm{i}\theta}z)=e^{\mathrm{i}\theta}F_1(z)$) for $n=2$. Ozawa and Zhai [25] was able to consider the