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On the variation of maximal operators of convolution type

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Abstract

In this paper we study the regularity properties of two maximal operators of convolution type: the heat flow maximal operator (associated to the Gauss kernel) and the Poisson maximal operator (associated to the Poisson kernel). In dimension $d = 1$ we prove that these maximal operators do not increase the L^p-variation of a function for any $p \geqslant 1$, while in dimensions $d > 1$ we obtain the corresponding results for the L^2-variation. Similar results are proved for the discrete versions of these operators.
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1. Introduction

1.1. Background

Let $\varphi \in L^1(\mathbb{R}^d)$ be a non-negative function such that

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x = 1.$$

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We let $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ and consider the maximal operator $\mathcal{M}_\varphi$ associated to this approximation of the identity

$$\mathcal{M}_\varphi f(x) = \sup_{t>0}\big(|f| * \varphi_t\big)(x). \tag{1.1}$$

The Hardy–Littlewood maximal function, henceforth denoted by M, occurs when we consider $\varphi(x) = (1/m(B_1))\chi_{B_1}(x)$, where B_1 is the d-dimensional ball centered at the origin with radius 1 and $m(B_1)$ is its Lebesgue measure. In a certain sense, one could say that M controls other such maximal operators of convolution type. In fact, if our φ admits a radial non-increasing majorant in $L^1(\mathbb{R}^d)$ with integral A, from [15, Chapter III, Theorem 2] we know that

$$\mathcal{M}_\varphi f(x) \leqslant AMf(x) \tag{1.2}$$

for all $x \in \mathbb{R}^d$ and thus we obtain the boundedness of $\mathcal{M}_\varphi$ from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ if $p > 1$, and from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ in the case $p = 1$.

Over the last years there has been considerable effort in understanding the effects of the Hardy–Littlewood maximal operator M, and some of its variants, in Sobolev functions. In [9] Kinnunen showed that $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is bounded for $p > 1$. The paradigm that an L^p-bound implies a $W^{1,p}$-bound was later extended to a local version of M in [10], to a fractional version in [11] and to a multilinear version in [5]. The continuity of $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ for $p > 1$ was established by Luiro in [13]. When $p = 1$ the issues become more subtle. The question on whether the operator $f \mapsto \nabla Mf$ is bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, posed by Hajłasz and Onninen in [7], remains open in its general case (see also [6]). Partial progress was achieved in the discrete setting in the work [3] for dimension $d = 1$ and in the work [4] for general dimension $d > 1$. In the continuous setting the only progress has been in dimension $d = 1$. For the right (or left) Hardy–Littlewood maximal operator, which we call here M_r (corresponding to $\varphi(x) = \chi_{[0,1]}(x)$ in (1.1)) Tanaka [17] was the first to observe that, if $f \in W^{1,1}(\mathbb{R})$, then $M_r f$ has a weak derivative and

$$\big\|(M_r f)'\big\|_{L^1(\mathbb{R})} \leqslant \big\|f'\big\|_{L^1(\mathbb{R})}, \tag{1.3}$$

which led to the bound for the *non-centered* Hardy–Littlewood maximal operator $\widetilde{M}$,

$$\big\|(\widetilde{M} f)'\big\|_{L^1(\mathbb{R})} \leqslant 2\big\|f'\big\|_{L^1(\mathbb{R})}. \tag{1.4}$$

This was later refined by Aldaz and Pérez-Lázaro [1] who obtained, under the assumption that f is of bounded variation on $\mathbb{R}$, that $\widetilde{M} f$ is absolutely continuous and

$$V(\widetilde{M} f) \leqslant V(f), \tag{1.5}$$

where $V(f)$ denotes here the total variation of f. More recently, in the remarkable work [12], Kurka showed that if f is of bounded variation on $\mathbb{R}$, then

$$V(Mf) \leqslant CV(f), \tag{1.6}$$

for a certain $C > 1$ (see [16] for the discrete analogue).

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