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Sub-additive ergodic theorems for countable amenable groups

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In this paper we generalize Kingman's sub-additive ergodic theorem to a large class of infinite countable discrete amenable group actions.

1. Introduction

The study of ergodic theorems was started in 1931 by von Neumann and Birkhoff, growing from problems in statistical mechanics. Ergodic theory soon earned its own place as an important part of functional analysis and probability, and grew into the study of measure-preserving transformations of a measure space. In 1968 an important new impetus to this area was received from Kingman's proof of the sub-additive ergodic theorem.

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This theorem opened up an impressive array of new applications [12–14]. Krengel [15] showed that Kingman's theorem can be used to derive the multiplicative ergodic theorem of Oseledec [24], which is of considerable current interest in the study of differentiable dynamical systems. Today, there are many elegant proofs of the theorem [5,11,13,20,27, 28]. Among these, perhaps the shortest proof is that of Steele [28]: this relies neither on a maximal inequality nor on a combinatorial Riesz lemma. A lovely exposition of the whole theory is given in Krengel's book [15].

A statement of Kingman's sub-additive ergodic theorem is as follows:

Theorem 1.1. *Let ϑ be a measure preserving transformation over the Lebesgue space $(Y, \mathcal{D}, \nu)$ and $\{f_n : n \in \mathbb{N}\} \subseteq L^1(Y, \mathcal{D}, \nu)$ satisfy $f_{n+m}(y) \leq f_n(y) + f_m(\vartheta^n y)$ for ν-a.e. $y \in Y$ and all $n, m \in \mathbb{N}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} f_n(y) = f(y) \geq -\infty$$

for ν-a.e. $y \in Y$, where f is an invariant measurable function over $(Y, \mathcal{D}, \nu)$.

If all the f_n are constant functions, equal to a_n (say), the theorem reduces to a well-known basic fact in analysis: if the sequence $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ satisfies $a_{n+m} \leq a_n + a_m$ for all $n, m \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n} \geq -\infty.$$

It is an easy extension of Kingman's theorem that if $\inf_n \frac{\int f_n d\nu}{n} > -\infty$, then the convergence also holds in L^1.

In this paper, we shall discuss extensions of Kingman's theorem to the class of countable discrete amenable groups.

The class of amenable groups includes all finite groups, solvable groups and compact groups, and actions of these groups on a Lebesgue space are a natural extension of the $\mathbb{Z}$-actions considered in Kingman's theorem: the foundations of the theory of amenable group actions were laid by Ornstein and Weiss in their pioneering paper [22].

Lindenstrauss [17] established the pointwise ergodic theorem for general locally compact amenable group actions along Følner sequences (with some natural conditions), which generalizes the Birkhoff theorem from $\mathbb{Z}$-actions to general amenable group actions, see also Benji Weiss' lovely survey article [33]. For other related work, see [1–3,7, 8,10,21,23,26,29,30].

In contrast to the amenable group $\mathbb{Z}$, a general infinite countable discrete amenable group may have a complicated combinatorial structure, and our challenge is to consider the limiting behavior of the Kingman type in this context.

In general, we define a subset $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ of functions in $L^1(Y, \mathcal{D}, \nu)$, indexed by the family $\mathcal{F}_G$ of all non-empty finite subsets of G, to be *G-invariant* and *sub-additive* if $d_{Eg}(y) = d_E(gy)$ and $d_{E\cup F}(y) \leq d_E(y) + d_F(y)$ for ν-a.e. $y \in Y$, any