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Nonlinear flows and rigidity results on compact manifolds

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A B S T R A C T

This paper is devoted to *rigidity* results for some elliptic PDEs and to *optimal constants* in related interpolation inequalities of Sobolev type on smooth compact connected Riemannian manifolds without boundaries. Rigidity means that the PDE has no other solution than the constant one at least when a parameter is in a certain range. The largest value of this parameter provides an estimate for the optimal constant in the corresponding interpolation inequality. Our approach relies on a nonlinear flow of porous medium / fast diffusion type which gives a clear-cut interpretation of technical choices of exponents done in earlier works on rigidity. We also establish two integral criteria for rigidity that improve upon known, pointwise conditions, and hold for general manifolds without positivity conditions on the curvature. Using the flow, we are also able to discuss the optimality of the corresponding constants in the interpolation inequalities.

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1. Introduction and main results

In the past decades there has been considerable activity in establishing sharp inequalities using maps or flows. The basic idea is to look for a flow on a function space along which a given functional converges to its optimal value, *i.e.*, one turns the idea of a Lyapunov function, known from dynamical systems theory, on its head. An example is furnished by the relatively recent proofs of the Brascamp–Lieb inequalities using nonlinear heat flows in [17,61,26]. Using the same methods a new Brascamp–Lieb type inequality on $\mathbb{S}^d$ was proved in [26]. Likewise the reverse Brascamp–Lieb inequalities can also be obtained in this fashion (see [14]). Another example is the proof of Lieb's sharp Hardy–Littlewood–Sobolev inequality given in [27] where a discrete map on a function space was constructed whose iterations drive the Hardy–Littlewood–Sobolev functional to its sharp value. Likewise, the sharp form of the Gagliardo–Nirenberg inequalities due to Dolbeault and del Pino can be derived using the porous media flow (see [29]). The porous media equation can also be used in the context of a special class of Hardy–Littlewood–Sobolev inequalities (see [25]).

Closely related are the proofs of sharp inequalities using transportation theory. The earliest use of transportation theory to our knowledge was in Barthe's proof of the Brascamp–Lieb inequalities as well as their converse, in [13]. Transportation ideas were also applied in [33] for proving the sharp Gagliardo–Nirenberg inequalities and in [57] for proving sharp trace inequalities.

In this paper we use a porous media flow on Riemannian manifolds that allow us to give relatively straightforward proofs as well as generalizations of rigidity results of [21,52,11,53] for a class of nonlinear equations. Before describing the flow, we discuss the rigidity results.

Throughout the paper we assume that $(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold of dimension $d \geq 1$, without boundary with Δ_g being the Laplace–Beltrami operator on $\mathfrak{M}$. For simplicity, we assume that the volume of $\mathfrak{M}$, $\mathrm{vol}(\mathfrak{M})$ is 1 and we denote by dv_g the volume element. We shall also denote by $\mathfrak{R}$ the Ricci tensor. Let λ_1 be the lowest positive eigenvalue of $-\Delta_g$. We shall use the notation $2^* := \frac{2d}{d-2}$ if $d \geq 3$, and $2^* := \infty$ if $d = 1$ or 2.

Let us start with results dealing with manifolds whose curvature is bounded from below and define

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi, \xi).$$

Theorem 1. *Let $d \geq 2$ be an integer and assume that ρ is positive. If λ is a positive parameter such that*

$$\lambda \leq (1-\theta)\lambda_1 + \theta \frac{d\rho}{d-1} \quad \textit{where } \theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1},$$

then for any $p \in (2, 2^*)$, the equation