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The full infinite dimensional moment problem on semi-algebraic sets of generalized functions

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ABSTRACT

We consider a generic basic semi-algebraic subset $\mathcal{S}$ of the space of generalized functions, that is a set given by (not necessarily countably many) polynomial constraints. We derive necessary and sufficient conditions for an infinite sequence of generalized functions to be *realizable* on $\mathcal{S}$, namely to be the moment sequence of a finite measure concentrated on $\mathcal{S}$. Our approach combines the classical results about the moment problem on nuclear spaces with the techniques recently developed to treat the moment problem on basic semi-algebraic sets of $\mathbb{R}^d$. In this way, we determine realizability conditions that can be more easily verified than the well-known Haviland type conditions. Our result completely characterizes the support of the realizing measure in terms of its moments. As concrete examples of semi-algebraic sets of generalized functions, we consider the set of all Radon measures and the set of all the measures having bounded Radon–Nikodym density w.r.t. the Lebesgue measure.

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1. Introduction

It is often more convenient to consider characteristics of a random distribution instead of the random distribution itself and try to extract information about the distribution from these characteristics. In this paper, we are more concretely interested in distributions on functional objects like random fields, random points, random sets and random measures. The characteristics under study are polynomials of these objects like the density, the pair distance distribution, the covering function, the contact distribution function, etc. This setting is considered in numerous areas of applications: heterogeneous materials and mesoscopic structures [44], stochastic geometry [29], liquid theory [14], spatial statistics [43], spatial ecology [30] and neural spike trains [7,16], just to name a few.

The subject of this paper is the full power moment problem on a pre-given subset $\mathcal{S}$ of $\mathscr{D}'(\mathbb{R}^d)$, the space of all generalized functions on $\mathbb{R}^d$. This framework choice is mathematically convenient and general enough to encompass all the aforementioned applications. More precisely, our paper addresses the question of whether certain prescribed generalized functions are in fact the moment functions of some finite measure concentrated on $\mathcal{S}$. If such a measure does exist, it will be called *realizing*. The main novelty of this paper is to investigate how one can read off support properties of the realizing measure directly from positivity properties of its moment functions.

To be more concrete, homogeneous polynomials are defined as powers of linear functionals on $\mathscr{D}'(\mathbb{R}^d)$ and their linear continuous extensions. We denote by $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'(\mathbb{R}^d))$ the set of all polynomials on $\mathscr{D}'(\mathbb{R}^d)$ with coefficients in $\mathcal{C}_c^\infty(\mathbb{R}^d)$, which is the set of all infinitely differentiable functions with compact support in $\mathbb{R}^d$.

In this paper, we try to find a characterization via moments of measures concentrated on basic *semi-algebraic* subsets of $\mathscr{D}'(\mathbb{R}^d)$, i.e. sets that are given by polynomial constraints and so are of the following form

$$\mathcal{S} = \bigcap_{i\in Y} \{\eta \in \mathscr{D}'(\mathbb{R}^d) \mid P_i(\eta) \geq 0\},$$

where Y is an arbitrary index set (not necessarily countable) and each P_i is a polynomial in $\mathscr{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'(\mathbb{R}^d))$. Equality constraints can be handled using P_i and $-P_i$ simultaneously. As far as we are aware, the infinite dimensional moment problem has only been treated in general on affine subsets [4,2] and cones [42] of nuclear spaces (these results are stated in Subsection 4.1 and Subsection 5.3). Special situations have also been handled; see e.g. [46,3,17].

1.1. Previous results

Characterization results via moments are built up out of five completely different types of conditions