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The Kato Square Root Problem for mixed boundary conditions ☆

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ABSTRACT

We consider the negative Laplacian subject to mixed boundary conditions on a bounded domain. We prove under very general geometric assumptions that slightly above the critical exponent $\frac{1}{2}$ its fractional power domains still coincide with suitable Sobolev spaces of optimal regularity. In combination with a reduction theorem recently obtained by the authors, this solves the Kato Square Root Problem for elliptic second order operators and systems in divergence form under the same geometric assumptions. Thereby we answer a question posed by J.L. Lions in 1962 [30].

1. Introduction

Let $-\nabla \cdot \mu \nabla$ be an elliptic differential operator in divergence form with bounded complex coefficients on a domain Ω, subject to Dirichlet boundary conditions on some closed subset D of the boundary $\partial\Omega$ and natural boundary conditions on $\partial\Omega \setminus D$ in

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the sense of the form method. Let A be the maximal accretive realization of $-\nabla \cdot \mu \nabla$ on $\mathrm{L}^2(\Omega)$. The *Kato Square Root Problem* for A, as posed by J.L. Lions in 1962 [30], amounts to identifying the domain of the maximal accretive square root of A as the domain of the corresponding form, i.e. the subspace of the first order Sobolev space $\mathrm{H}^1(\Omega)$ whose elements vanish on D. In this case A is said to have the *square root property.*

Whereas for self-adjoint A the square root property is immediate from abstract form theory [26], the problem for non-self-adjoint operators remained open for almost 40 years. For a historical survey explaining also the special role of the square root of A compared to other fractional powers, we refer to [3,33]. Shortly after being solved on the whole space by Auscher, Hofmann, Lacey, M^{c}Intosh, and Tchamitchian [3,4], Auscher and Tchamitchian used localization techniques to give a proof on strongly Lipschitz domains complemented by either pure Dirichlet or pure Neumann boundary conditions [5]. Earlier efforts concerning mixed boundary conditions culminated in the work of Axelsson, Keith, and M^{c}Intosh [6], who gave a proof for smooth domains with a Dirichlet part whose complement within the boundary is smooth and in addition – due to the first order structure of the problem – for global bi-Lipschitz images of these configurations.

The purpose of the present paper is to solve the Kato Square Root Problem on bounded domains under much more general geometric assumptions than in [5] and [6]. First and foremost we can dispense with the Lipschitz property of Ω in the following spirit: We assume that $\partial\Omega$ decomposes into a closed subset D, to be understood as the Dirichlet part, and its complement, which are allowed to share a common frontier. We demand that D is a $(d-1)$-set in the sense of Jonsson–Wallin, or equivalently satisfies the Ahlfors–David condition, and only around $\overline{\partial\Omega \setminus D}$ do we demand local bi-Lipschitz charts. In addition, we in essence impose a plumpness, or interior corkscrew condition on Ω, which, roughly speaking, excludes outward cusps also along the Dirichlet part. For precise definitions we refer to Section 2.

In particular, Ω may be sliced or touch its boundary from two sides, see Fig. 1 for a striking constellation.

As special cases the pure Dirichlet ($D = \partial\Omega$) and the pure Neumann case ($D = \emptyset$) are included. Let us stress that in the former we can dispense with the Lipschitz property of the domain completely.

More recently, relative results including the square root property for A as an assumption have been obtained. This concerns extrapolation of the square root property to L^p spaces [2], maximal parabolic regularity on distribution spaces [2], and perturbation theory [14]. One of our main motivations for writing the present paper was to close this gap between geometric constellations in which the Kato Square Root Problem is solved and those in which its solution already applies to other topics.

It is convenient to view the Kato Square Root Problem as the problem of proving optimal Sobolev regularity for the domain of the square root of A. Indeed, as A is associated to a second order differential operator, the domain of A allows for at most two distributional derivatives in $\mathrm{L}^2(\Omega)$. Hence, by interpolation the optimal Sobolev regularity for the domain of its square root is one distributional derivative in $\mathrm{L}^2(\Omega)$. It is