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Global well-posedness of the two-dimensional incompressible magnetohydrodynamics system with variable density and electrical conductivity

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Studied in this paper is the Cauchy problem of the two-dimensional magnetohydrodynamics system with inhomogeneous density and electrical conductivity. It is shown that the 2-D incompressible inhomogeneous magnetohydrodynamics system with a constant viscosity is globally well-posed for a generic family of the variations of the initial data and an inhomogeneous electrical conductivity. Moreover, it is established that the system is globally well-posed in the critical spaces if the electrical conductivity is homogeneous.

1. Introduction

This paper is concerned with the global well-posedness of the following incompressible inhomogeneous magnetohydrodynamics (MHD) equations with initial data in the critical Besov spaces

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$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\big(2\mu(\rho)\mathcal{M}\big) + \nabla \Pi = B \cdot \nabla B, \\ \partial_t B + u \cdot \nabla B + \operatorname{curl}\left(\dfrac{\operatorname{curl} B}{\sigma(\rho)}\right) = B \cdot \nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ \rho|_{t=0} = \rho_0, \qquad u|_{t=0} = u_0, \qquad B|_{t=0} = B_0, \end{cases} \tag{1.1}$$

where the unknowns are the density ρ, the velocity $u = (u_1, u_2, u_3)$, the magnetic field $B = (B_1, B_2, B_3)$, and the pressure Π. We denote here by $\mathcal{M} = \frac{1}{2}(\nabla u + \nabla u^T)$ the deformation tensor, σ the electrical conductivity of the field and μ its viscosity, both conductivity and viscosity being smooth, nonnegative functions of the density $\sigma = \sigma(\rho)$, $\mu = \mu(\rho)$, $\operatorname{curl} B \stackrel{\text{def}}{=} (\partial_2 B_3 - \partial_3 B_2, \partial_3 B_1 - \partial_1 B_3, \partial_1 B_2 - \partial_2 B_1)^T$.

The MHD system (1.1) is a combination of the incompressible inhomogeneous Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism, where the displacement current can be neglected [28,29]. Such a system describes the motion of several conducting incompressible immiscible fluids without surface tension in presence of a magnetic field [25].

There have been a lot of studies on magnetohydrodynamics by physicists and mathematicians because of their prominent roles in modeling many phenomena in astrophysics, geophysics and plasma physics, see [3,5,9,13,22,23,25–27,30,34] and the references cited therein.

For the homogeneous fluid (i.e. the density ρ is a positive constant), the MHD system has been extensively studied. For instance, G. Duvaut and J.-L. Lions [23] established the local existence and uniqueness of a solution in the classical Sobolev spaces $H^s(\mathbb{R}^d)$, $s \geq d$, and proved global existence of the solution for small initial data. M. Sermange and R. Temam further examined the properties of these solutions [34]. In particular, the 2D local strong solution has been proved to be global and unique. Recent work on the homogeneous MHD equations developed regularity criteria in terms of the velocity field and dealt with the MHD equations with dissipation and magnetic diffusion (see, e.g., [13,26]). Also the issue of the global regularity problem on the MHD equations with partial dissipation, especially in the 2D case, has been extensively studied (see, e.g., [9,30]).

When the fluid is inhomogeneous, it has been also studied by many authors. It is known that J.-F. Gerbeau and C. Le Bris [25] (see also B. Desjardins and C. Le Bris [22]) established the global existence of weak solutions of finite energy in the whole space $\mathbb{R}^3$ or in the torus $\mathbb{T}^3$. Chen, Tan, and Wang [14] demonstrated the local strong solutions to (1.1) in presence of vacuum under the assumptions that both conductivity and viscosity are constants. On the other hand, Huang and Wang [27] extended it to the global existence of strong solutions.

If the density ρ is away from zero, let us denote $a \stackrel{\text{def}}{=} \rho^{-1} - 1$, $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\rho)$, and $\sigma_1(a) \stackrel{\text{def}}{=} \frac{1}{\sigma(\rho)}$. Then the system (1.1) can be equivalently reformulated as