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MF actions and K-theoretic dynamics

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ABSTRACT

We study the interplay of C^*-dynamics and K-theory. Notions of chain recurrence for transformations groups (X, Γ) and MF actions for non-commutative C^*-dynamical systems (A, Γ, α) are translated into K-theoretical language, where purely algebraic conditions are shown to be necessary and sufficient for a reduced crossed product to admit norm microstates. We are particularly interested in actions of free groups on AF algebras, in which case we prove that a K-theoretic coboundary condition determines whether or not the reduced crossed product is a matricial field (MF) algebra. One upshot is the equivalence of stable finiteness and being MF for these reduced crossed product algebras.

1. Introduction

The theories of operator algebras and dynamical systems are closely interwoven. Crossed product algebras arising from transformation groups, and in general from non-commutative C^*-dynamical systems, play an important role in the study of C^*-algebras. One would like to uncover information about the algebra by studying the dynamics and, conversely, describe the nature of the dynamics by looking at the operator algebra's structure and invariants.

We are particularly interested in finite-dimensional approximation properties of C^*-algebras. While nuclearity and exactness are measure-theoretic concepts, residual finite dimensionality, quasidiagonality and admitting norm microstates are properties

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more topological in nature as they concern matricial approximation of both the linear and multiplicative structure of the algebra. In this work we flesh out the appropriate dynamical conditions that give rise to such topological approximations in resulting reduced crossed products, and give K-theoretic expression to these conditions when the underlying algebras have sufficiently many projections. One purpose of this paper is to provide a K-theoretic interpretation of dynamical approximation properties such as residual finiteness and quasidiagonal actions as introduced by Kerr and Nowak in [14], and by doing so, extend results found in [3] and [17].

In the classical setting, Pimsner described a purely topological dynamical property for a $\mathbb{Z}$-system $(X, \mathbb{Z})$ that renders the resulting crossed product $C(X) \rtimes_\lambda \mathbb{Z}$ AF-embeddable. He showed in [17] that for a self-homeomorphism T of a compact metrizable space X the following are equivalent: (1) the crossed product embeds into an AF algebra, (2) the crossed product is quasidiagonal, (3) the crossed product is stably finite, (4) "T compresses no open sets", that is, there does not exist a non-empty open set $U \subset X$ for which $T(\overline{U}) \subsetneq U$, which is equivalent to the action being *chain recurrent*, that is, for every $x \in X$ and every $\varepsilon > 0$, there are finitely many points $x = x_1, \dots, x_n = x$ such that $d(T(x_j), x_{j+1}) < \varepsilon$ for $1 \le j \le n$.

It was N. Brown who saw condition (4) as being essentially K-theoretical, at least in the presence of many projections [3]. When X is zero-dimensional we have $K_0(C(X)) = C(X; \mathbb{Z})$, and the chain recurrence condition is expressed as $\hat{T}(f) < f$ for no non-zero $f \in C(X; \mathbb{Z})$, where $\hat{T} : C(X; \mathbb{Z}) \to C(X; \mathbb{Z})$ is the induced order automorphism given by $\hat{T}(f) = f \circ T^{-1}$. Brown was then able to generalize Pimsner's result to the noncommutative setting as follows.

Theorem 1.1 *(Brown). Let A be an AF algebra and $\alpha \in \mathrm{Aut}(A)$ an automorphism. Then the following are equivalent:*

(1) *$A \rtimes_\alpha \mathbb{Z}$ is AF-embeddable.*
(2) *$A \rtimes_\alpha \mathbb{Z}$ is quasidiagonal.*
(3) *$A \rtimes_\alpha \mathbb{Z}$ stably finite.*
(4) *The induced map $\hat{\alpha} : K_0(A) \to K_0(A)$ "compresses no element", that is, there is no $x \in K_0(A)$ for which $\hat{\alpha}(x) < x$; equivalently, $H_\alpha \cap K_0(A)^+ = \{0\}$, where H_α is the coboundary subgroup $\{x - \hat{\alpha}(x) : x \in K_0(A)\}$.*

One of the main results of this paper, Theorem 4.14, extends Brown's result to the case of a free group on r generators acting on a unital AF algebra. In this case the coboundary subgroup is given by $H_\sigma = \mathrm{im}(\sigma)$ where $\sigma : \bigoplus_{j=1}^r K_0(A) \to K_0(A)$ is the coboundary morphism in the Pimsner–Voiculescu six-term exact sequence. In abbreviated form our theorem says the following.

Theorem 1.2. *Let A be a unital AF algebra and $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ an action of the free group on r generators. Then the following are equivalent:*