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Global weighted estimates for quasilinear elliptic equations with non-standard growth

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ABSTRACT

In this paper we obtain a new global gradient estimates in weighted Lorentz spaces for weak solutions of $p(x)$-Laplacian type equation with small BMO coefficients in a δ-Reifenberg flat domain. The modified Vitali covering lemma, the maximal function technique and the appropriate localization method are the main analytical tools. Our results improve the known results for such equations.

1. Introduction

Let Ω be an open, bounded domain in $\mathbb{R}^N$. In this paper we are concerned with the weighted gradient estimates of weak solutions for the following quasilinear elliptic equation:

$$\operatorname{div}\Big((A\nabla u\cdot\nabla u)^{\frac{p(x)-2}{2}}A\nabla u\Big)=\operatorname{div}\Big(|\mathbf{f}|^{p(x)-2}\mathbf{f}\Big)\quad\text{in }\Omega. \tag{1.1}$$

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Here the variable exponent $p:\overline{\Omega} \to (1,\infty)$ is a continuous function, $\mathbf{f}$ is a given vector valued function satisfying that $|\mathbf{f}|^{p(x)}$ at least belongs to $L^1(\Omega)$, and $A = [a_{ij}(x)]_{N\times N}$ is a symmetric matrix with measurable coefficients satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leqslant A(x)\xi \cdot \xi \leqslant \Lambda|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$, where λ and Λ are positive constants.

The study of partial differential equations and variational problems with non-standard growth conditions has been received considerable attention by many models coming from various branches of mathematical physics, such as elastic mechanics, image processing and electro-rheological fluid dynamics, etc. We refer the readers to [1,10,26,27] and references therein. Elliptic equations of the type considered in (1.1) are simplified versions of equations which arise naturally in the mathematical modeling of electro-rheological fluids developed by Rajagopal and Růžička [26]. These are particular non-Newtonian fluids, characterized by their ability of changing their mechanical properties when interacting with an electromagnetic field $\mathbf{E}(x)$. The viscosity strongly depends on external electromagnetic fields and therefore varies in space and time. The model for the steady case is similar to the modified Navier–Stokes system, i.e.,

$$-\operatorname{div} S\big(x, \varepsilon(v)\big) = g(x, v, \nabla v), \qquad \operatorname{div} v = 0,$$

where v is the velocity of the fluid, $\varepsilon(v)$ is the symmetric part of the gradient ∇v and the tensor S satisfies the standard monotonicity conditions in Leray–Lions class but with $p(x)$-growth in the sense that

$$D^2 S(x,z) \geqslant \nu\big(1 + |z|^2\big)^{(p(x)-2)/2}\mathbf{Id},$$

where $p(x) \equiv p(|\mathbf{E}|^2)$ is a function of $\mathbf{E}$ which is given. At the same time, other models of this type arise for fluids whose viscosity is influenced in a similar way by the temperature (see [33]). The differential system modeling the so called thermistor problem (see [31–33]) includes the equations like

$$-\operatorname{div}\big(p(x)|\nabla u|^{p(x)-2}\nabla u\big) = 0.$$

To introduce the main results we need to state some standard notation. Denote $B_R(x)$ as a ball in $\mathbb{R}^N$, centered at the point x, with radius $R > 0$. The integral average of an integral function $f \in L^1(E)$ on a measurable subset E of $\mathbb{R}^N$ is defined by

$$\bar{f}_E = \fint_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx,$$

where $|E|$ is the Lebesgue measure of set E.

In this paper, we are mainly interested in finding a minimal requirement on the matrix of coefficient A and a lower level of geometric assumption on the boundary $\partial\Omega$ as well as the