Journal of Functional Analysis 266 (2014) 6319–6334

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Boundary C^*-algebras of triangle geometries

Guyan Robertson

School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, England, UK

ARTICLE INFO

Article history:
Received 9 September 2011
Accepted 19 February 2014
Available online 27 March 2014
Communicated by D. Voiculescu

Keywords:
Euclidean building
Maximal boundary
C^*-algebra

ABSTRACT

Let Δ be a building of type $\widetilde{A}_2$ and order q, with maximal boundary Ω. Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ. Then the crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$ is isomorphic to $M_{3(q+1)} \otimes \mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$, where $\mathcal{O}_n$ denotes the Cuntz algebra generated by n isometries whose range projections sum to the identity operator.

1. Introduction

The boundary at infinity plays an important role in studying the action of groups on euclidean buildings and other spaces of non-positive curvature. The motivation of this article is to determine what information about a group may be recovered from the K-theory of the boundary crossed product C^*-algebra. The focus is on groups of automorphisms of euclidean buildings. An action of a group on a set is said to be *regular* if it is both free and transitive. Groups of automorphisms of $\widetilde{A}_2$ buildings which act regularly on the set of chambers have been studied by several authors [6,7,12,14]. The existence of groups which act regularly on the set of chambers (i.e. the existence of "tight triangle geometries") is proved in [12, Theorem 3.3].

E-mail address: guyanrobertson@gmx.com.

http://dx.doi.org/10.1016/j.jfa.2014.02.031
0022-1236/

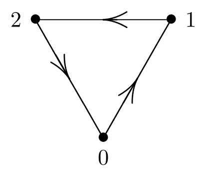

Fig. 1. A chamber showing vertex types and edge orientations.

The main result is the following, where M_n denotes the algebra of complex $n \times n$ matrices and $\mathcal{O}_n$ is the Cuntz C^*-algebra which is generated by n isometries on a Hilbert space whose range projections sum to I.

Theorem 1.1. *Let Δ be a building of type $\widetilde{A}_2$ and order q, with maximal boundary Ω. Let Γ be a group of type preserving automorphisms of Δ which acts regularly on the chambers of Δ. Then the crossed product C^*-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$ is isomorphic to $M_{3(q+1)} \otimes \mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$.*

The theorem is proved by giving an explicit representation of $\mathfrak{A}_\Gamma$ as a rank two Cuntz–Krieger algebra in the sense of [10]. The K-theory of $\mathfrak{A}_\Gamma$ is then determined explicitly as $K_0(\mathfrak{A}_\Gamma) \cong K_1(\mathfrak{A}_\Gamma) \cong \mathbb{Z}_{q^2-1}$ with the class $[\mathbf{1}]$ in $K_0(\mathfrak{A}_\Gamma)$ corresponding to $3(q+1) \in \mathbb{Z}_{q^2-1}$. The classification theorem of E. Kirchberg and N.C. Phillips for p.i.s.u.n. C^* algebras [1] completes the proof. It is notable that, for these groups, $C(\Omega) \rtimes \Gamma$ depends only on the order q of the building and not on the group Γ. This is the first class of higher rank groups for which the boundary C^*-algebra has been computed exactly. There is a sharp contrast with the class of groups which act regularly on the vertex set of an $\widetilde{A}_2$ building, where the boundary C^*-algebra $\mathfrak{A}_\Gamma$ frequently appears to determine the group Γ, according to explicit computations derived from [11]. For example, the three torsion-free groups $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$ which act regularly on the vertex set of the corresponding building Δ are distinguished from each other by $K_0(\mathfrak{A}_\Gamma)$. On the other hand, for $q = 2$, there are four different groups satisfying the hypotheses of Theorem 1.1, and hence having the same boundary C^*-algebra.

2. Triangle geometries

A locally finite euclidean building whose boundary at infinity is a spherical building of rank 2 is of type $\widetilde{A}_2$, $\widetilde{B}_2$ or $\widetilde{G}_2$. This article concerns buildings of type $\widetilde{A}_2$, also called triangle buildings. There is a close connection to projective geometry: the link of each vertex of an $\widetilde{A}_2$ building Δ is the incidence graph of a finite projective plane of order q. The maximal simplices of Δ are triangles, which are called *chambers*. Each vertex of Δ has a *type* $j \in \mathbb{Z}_3$, and each chamber has exactly one vertex of each type. Each edge of Δ lies on $q+1$ chambers. Orient each edge of Δ from its vertex of type i to its vertex of type $i+1$ (see Fig. 1). In the link of the vertex v of type i, the vertices of type $i+1$ [type $i+2$] correspond to the points [lines] of a projective plane of order q. The chambers of Δ