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A reduction method for semilinear elliptic equations and solutions concentrating on spheres ☆

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ABSTRACT

We show that any general semilinear elliptic problem with Dirichlet or Neumann boundary conditions in an annulus $A \subseteq \mathbb{R}^{2m}$, $m \geqslant 2$, invariant by the action of a certain symmetry group can be reduced to a nonhomogeneous similar problem in an annulus $D \subset \mathbb{R}^{m+1}$, invariant by another related symmetry. We apply this result to prove the existence of positive and sign changing solutions of a singularly perturbed elliptic problem in A which concentrate on one or two $(m-1)$ dimensional spheres. We also prove that the Morse indices of these solutions tend to infinity as the parameter of concentration tends to infinity.

1. Introduction

In this paper we propose a method to reduce a semilinear elliptic problem of the type:

$$\begin{cases} -\Delta u = f(u) & \text{in } A \subseteq \mathbb{R}^{2m} \\ u = 0 \text{ or } \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial A \end{cases} \tag{1.1}$$

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where A is an annulus in $\mathbb{R}^{2m}$, $m \geqslant 2$,

$$A = \{x \in \mathbb{R}^{2m}:\ a < |x| < b,\ 0 < a < b\}$$

and f is a $C^{1,\alpha}$ nonlinearity, to the semilinear elliptic problem:

$$\begin{cases} -\Delta v = \dfrac{f(v)}{2|z|} & \text{in } D \subseteq \mathbb{R}^{m+1} \\ v = 0 \text{ or } \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial D \end{cases} \tag{1.2}$$

where $z \in \mathbb{R}^{m+1}$ and D is the annulus

$$D = \left\{z \in \mathbb{R}^{m+1}:\ \frac{a^2}{2} < |z| < \frac{b^2}{2}\right\}$$

As will be clear from the construction, there will be a one to one correspondence between solutions of (1.1) invariant under the action of a symmetry group in $H_0^1(A)$ or $H^1(A)$ and solutions of (1.2) invariant under another symmetry in $H_0^1(D)$ or $H^1(D)$.

More precisely, writing $x \in \mathbb{R}^{2m}$ as $x = (y_1, y_2)$, $y_i \in \mathbb{R}^m$, $i = 1, 2$, we consider solutions u of (1.1) which are radially symmetric in y_1 and y_2 i.e. $u(x) = w(|y_1|, |y_2|)$ and solutions v of (1.2) which are axially symmetric i.e. $v(z) = h(|z|, \varphi)$ with $\varphi = \arccos(\frac{z}{|z|} \cdot p)$ for a unit vector $p \in \mathbb{R}^{m+1}$.

Since the domains are annuli, by standard regularity theory all solutions we consider are classical $C^{2,\alpha}$-solutions. We set:

$$X = \{u \in C^{2,\alpha}(\overline{A}):\ u(x) = w(|y_1|, |y_2|)\}$$
$$Y = \{v \in C^{2,\alpha}(\overline{D}):\ v \text{ axially symmetric}\}$$

Our result is the following:

Theorem 1.1. *There is a bijective correspondence between solutions of (1.1) in X and solutions of (1.2) in Y.*

The map which gives the bijection to prove Theorem 1.1 will be defined in Section 3 after choosing suitable coordinates in $\mathbb{R}^{2m}$ and $\mathbb{R}^{m+1}$.

The possibility of reducing a problem in dimension $2m$ to a problem in the lower dimension $(m+1)$ is of great importance in the study of semilinear elliptic equations. As example one can think of the case of power nonlinearities when critical or supercritical problems in $\mathbb{R}^{2m}$ can become subcritical in $\mathbb{R}^{m+1}$. Moreover solutions concentrating on sets of a certain dimension in $\mathbb{R}^{m+1}$ (e.g. points) can give rise to solutions concentrating on higher dimensional manifolds on $\mathbb{R}^{2m}$.

Indeed, the inspiration for our method came from the paper [14] where a reduction method was introduced to pass from a singularly perturbed problem in an annulus in $\mathbb{R}^4$