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Weak and cyclic amenability for Fourier algebras of connected Lie groups

Yemon Choi *, Mahya Ghandehari [1]

Department of Mathematics and Statistics, McLean Hall, University of Saskatchewan, Saskatoon (SK), S7N 5E6 Canada

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ABSTRACT

Using techniques of non-abelian harmonic analysis, we construct an explicit, non-zero cyclic derivation on the Fourier algebra of the real $ax+b$ group. In particular this provides the first proof that this algebra is not weakly amenable. Using the structure theory of Lie groups, we deduce that the Fourier algebras of connected, semisimple Lie groups also support non-zero, cyclic derivations and are likewise not weakly amenable. Our results complement earlier work of Johnson (1994) [15], Plymen (2001) [18] and Forrest, Samei, and Spronk (2009) [9]. As an additional illustration of our techniques, we construct an explicit, non-zero cyclic derivation on the Fourier algebra of the reduced Heisenberg group, providing the first example of a connected nilpotent group whose Fourier algebra is not weakly amenable.

* Corresponding author. Current address: Department of Mathematics and Statistics, Fylde College, Lancaster University, Lancaster, LA1 4YF, United Kingdom.

E-mail addresses: y.choi1@lancaster.ac.uk (Y. Choi), mghandeh@fields.utoronto.ca (M. Ghandehari).

[1] Current address: The Fields Institute for Research in Mathematical Sciences, 222 College Street, Toronto, Ontario M5T 3J1, Canada.

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1. Introduction

1.1. Background context and history

The study of derivations from Banach algebras into Banach bimodules has a long history. In many cases, where the algebra consists of functions on some manifold or well-behaved subset of Euclidean space, and the target bimodule is symmetric, continuous derivations can be constructed by taking weighted averages of derivatives of functions. This leads to examples where the algebra does not admit any non-zero continuous 'point' derivations, yet admits a non-zero continuous derivation into *some* symmetric Banach bimodule; this is often a manifestation of some kind of vestigial analytic structure or Hölder continuity of functions in the algebra. Commutative Banach algebras which admit no non-zero, continuous derivations into *any* symmetric Banach bimodule are said to be *weakly amenable*: the terminology was introduced by W.G. Bade, P.C. Curtis Jr. and H.G. Dales in [2], where they studied some key examples in detail.

One natural class of function algebras not considered in [2] is the class of Fourier algebras of locally compact groups, first defined in full generality by P. Eymard in [5]. Fourier algebras never admit any non-zero, continuous point derivations: this follows from e.g. [5, (4.11)]. Moreover, if G is a locally compact abelian group, its Fourier algebra $\mathrm{A}(G)$ is isomorphic via the classical Fourier transform to the convolution algebra $L^1(\widehat{G})$, and so by results of B.E. Johnson it is *amenable*, hence weakly amenable (see [14, Proposition 8.2]).

Now a recurring theme in abstract harmonic analysis, and the study of Fourier algebras in particular, is the hope that known results for locally compact abelian groups can be generalized in a natural way to the class of locally compact amenable groups. It was therefore something of a surprise when Johnson, in [15], constructed a non-zero bounded derivation from $\mathrm{A}(\mathrm{SO}_3(\mathbb{R}))$ to its dual. Subsequently, using the structure theory of semisimple Lie algebras, R.J. Plymen observed [18] that Johnson's construction can be transferred to yield non-zero continuous derivations on $\mathrm{A}(G)$ for any non-abelian, compact, connected Lie group. This was extended further by B.E. Forrest, E. Samei and N. Spronk in [9], who showed using structure theory for compact groups that Plymen's result remains valid if one drops the word "Lie".

The non-compact, connected case has received relatively little attention. The articles [18] and [9] ultimately rely on locating closed copies of $\mathrm{SO}_3(\mathbb{R})$ or $\mathrm{SU}_2(\mathbb{C})$ inside the group in question, and then transporting Johnson's derivation along the corresponding restriction homomorphism of Fourier algebras. Indeed, as far as the present authors are aware, all results to date which show that $\mathrm{A}(G)$ fails to be weakly amenable only work for those G containing compact, connected, non-abelian subgroups. This has left open several natural examples, such as the "real $ax + b$ group" (to be defined precisely in Section 4), or certain semisimple Lie groups such as $\mathrm{SL}_2(\mathbb{R})$ and its covering groups.